Pergamon

Int. J. Solids Structures Vol. 35, No. 16, pp. 1915–1952, 1998
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0020–7683/98 $19.00 + .00

PII: S0020–7683(97)00130–3

COUPLED DEFORMATIONS OF ELASTIC CURVES AND SURFACES

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(*Received* 10 *December* 1996; *in revised form* 1 *May* 1997)

Abstract—An equilibrium theory for the coupled finite deformations of elastic curves and surfaces is described. Possible wrinkling of the curve or surface is taken into account by using a *relaxed* version of the theory obtained from minimum energy considerations. The relaxed theory admits a dual formulation leading to extremum principles and uniqueness of the equilibrium stress distribution. A number of examples are treated using spatial finite differences together with a dynamic relaxation method in which equilibrium configurations are obtained in the long-time limit of a damped dynamical problem. © 1998 Elsevier Science Ltd.

1. INTRODUCTION

Theories of perfectly flexible one- and two-dimensional elastic continua have long been used to model diverse phenomena ranging from bioelasticity and fluid capillarity to rubber elasticity and the mechanics of structural networks. It is rare, however, to find treatments of these subjects that take three-dimensional interactions of the two types of continua into account in a setting general enough to allow for finite deformations and strains. Our aim, in this work, is to present such a framework and to outline an associated body of theory sufficient to support the analysis of a range of practical problems. Most of our results bear directly on the mechanics of light weight tension structures, and a number are potentially applicable to biomechanics.

Our main interest here is the use of ideal models of perfectly flexible continua to solve boundary value problems in which compressive stresses would normally be expected to appear in various parts of the structure. From the standpoint of the more sophisticated theories of rods and shells, such stress states, if sufficiently intense, would be expected to lead to bifurcation and possible instability. Empirically, the post-bifurcation equilibrium response typically involves wrinkling under small compressive stress with most of the (tensile) stress transmitted along the wrinkle trajectories, as in the post-buckling of a thin shear panel, for example. The magnitude of compressive stress and the wavelength and amplitude of the wrinkles are determined in large part by the flexural stiffness of the material.

In principle, such a pattern of deformation cannot be described by the ideal models due to the total absence of flexural resistance. However, the use of theories with structure sufficient to describe the detailed configurations of wrinkle patterns entails substantial additional analytical or computational effort. Often the main features of interest, such as the global force–deflection response, are not particularly sensitive to this kind of local detail. Thus, we base our analysis on the so-called *relaxed* theories of perfectly flexible continua, in which the wrinkling associated with incipient compression is accommodated by modifying the constitutive equations in such a way that compressive stress is automatically excluded at all strains. The deformation corresponding to a state of strain that would generate destabilizing compression according to the original theory may instead be viewed as resulting from a continuous distribution of wrinkles under negligible compressive stress. Pipkin (1986) showed that such a model can be reconciled with the conventional theory of elastic surfaces by considering minimizing sequences for an associated variational problem. These sequences possess a structure similar to the wrinkling observed in actual thin elastic sheets.

We begin in Section 2 with brief accounts of the conventional theories of elastic curves and surfaces. We shall often refer to these as cable and membrane theories in accordance with common practice. A variational framework for the coupled response of curves and surfaces is discussed in Section 3. The variational approach is adopted here to facilitate contact with earlier work on the relaxation of membrane theory and to provide a rational basis for extension to the present class of problems. We derive equilibrium conditions for two classes of elastic interaction. In the first, the cable is regarded as being embedded in the membrane along its edge, while in the second the cable and membrane are free to slide relative to each other without friction. Further restrictions on the stress associated with an energy minimizing configuration are obtained. These, in turn, motivate the construction and interpretation of the relaxed version of the theory.

Under certain conditions the relaxed theory admits a dual variational formulation that can be used to generate global extremum principles and to establish uniqueness of some of the features of equilibrium states. These concepts are described in Section 4, where we prove a theorem of minimum complementary potential energy for cable-membrane interactions. In Section 5 we discuss a numerical method based on spatial finite differences and a *dynamic relaxation* scheme for computing equilibrium configurations. In this method the discretized equilibrium equations are replaced by an artificially damped finite-dimensional dynamical problem and equilibria are recovered asymptotically in the large time limit. This effectively regularizes the ill-conditioning of conventional stiffness-based methods for the class of problems considered. The modifications required to incorporate cable-membrane interaction are described and a number of example problems illustrating various types of coupling are discussed in Section 6.

2. ELASTIC CURVES AND SURFACES

In this section we briefly recount those elements of the theories of perfectly flexible elastic curves and surfaces that are needed in this work. Fuller accounts are given by Atai and Steigmann (1997) and by Haseganu and Steigmann (1994a).

2.1. *Elastic curves*

One may construct a theory for perfectly flexible elastic curves, or cables, on the postulate that there exists a strain energy, B, per unit length of arc of the curve in a reference placement, that depends in a specified way on the values of $\mathbf{r}'(s)$. Here $\mathbf{r}$ is the position vector of a material point of the curve in a typical configuration, s measures arclength in a reference configuration, and the prime is used to denote differentiation with respect to the indicated argument. Non-uniform materials may be taken into account by allowing B to depend explicitly on s, in addition to $\mathbf{r}'$. We suppress reference to explicit s-dependence here.

For $B(\mathbf{r}')$ to meet the usual restriction of invariance under superposed rigid motions, it is necessary and sufficient that

$$B(\mathbf{r}') = \hat{B}(\lambda)\,; \quad \lambda(s) = |\mathbf{r}'(s)|, \tag{1}$$

where λ is the local stretch of the curve. Unstretched configurations are defined by $\lambda(s) = 1$. Thus, the strain energy stored in an elastic curve of reference length L is:

$$\int_0^L B(\mathbf{r}'(s))\,ds = \int_0^L \hat{B}(\lambda(s))\,ds. \tag{2}$$

The force exerted by the part $(s, L]$ of the curve on the part $[0, s]$ is (Atai and Steigmann, 1997):

$$\mathbf{f}(s) = \hat{\mathbf{f}}(\mathbf{r}'(s)) = (\partial B/\partial r_i')\mathbf{e}_i, \tag{3}$$

where $\{\mathbf{e}_i\}$; $i = 1, 2, 3$, is a fixed orthonormal basis and $r_i(s)$ are the associated components of $\mathbf{r}(s)$. The force exerted by $[0, s)$ on $[s, L]$ is $-\mathbf{f}(s)$. It follows from eqn (1) that

$$\partial B/\partial r_i' = f(\lambda)t_i, \tag{4}$$

where

$$f(\lambda) = \hat{B}'(\lambda) \tag{5}$$

and

$$\mathbf{t}(s) = \lambda^{-1}\mathbf{r}'(s) \tag{6}$$

is the unit tangent to the deformed curve. Thus

$$\mathbf{f}(s) = f(\lambda(s))\mathbf{t}(s). \tag{7}$$

Evidently the force is everywhere tangential to the space curve defined by the function $\mathbf{r}(\cdot)$. The constitutive response is given by the function $f(\cdot)$.

If the curve is in equilibrium with a distributed force $\mathbf{b}(s)$ per unit reference arclength, then

$$\mathbf{f}'(s) + \mathbf{b}(s) = \mathbf{0}, \quad s \in (0, L). \tag{8}$$

If, in addition, end forces $\mathbf{f}_L$ and $-\mathbf{f}_0$ are applied at $s = L$ and $s = 0$, respectively, then

$$\mathbf{f}(0) = \mathbf{f}_0 \quad \text{and} \quad \mathbf{f}(L) = \mathbf{f}_L. \tag{9}$$

In this case $\mathbf{f}_0$, $\mathbf{f}_L$ and $\mathbf{b}(s)$ must be chosen such that

$$\mathbf{f}_L = \mathbf{f}_0 - \int_0^L \mathbf{b}(s)\,ds. \tag{10}$$

Alternatively, if either of the ends of the curve is constrained against movement then the relevant equality in eqn (9) defines the operative reaction force at the end in question, and eqn (10) is automatically satisfied in any equilibrium configuration.

2.2. *Elastic surfaces*

A two-dimensional generalization of the foregoing theory furnishes a model suitable for the analysis of perfectly flexible elastic surfaces (membranes). For our present purposes it suffices to consider a flat reference surface that occupies a bounded region Ω of the (x_1, x_2)—plane with piecewise smooth boundary $\partial\Omega$. Particles of the surface are identified by their position vectors $\mathbf{x} = x_\alpha \mathbf{e}_\alpha$, where Greek indices range over $\{1, 2\}$ and $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a fixed orthonormal basis that spans Ω.

A three-dimensional deformation of the surface is described in parametric form by the mapping $\mathbf{x} \to \mathbf{y}(\mathbf{x}) = y_i(\mathbf{x})\mathbf{e}_i$, where Latin indices take values in $\{1, 2, 3\}$ and $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$. The associated deformation gradient is

$$\mathbf{F}(\mathbf{x}) = \operatorname{grad} \mathbf{y}(\mathbf{x}) = F_{i\alpha}(\mathbf{x})\mathbf{e}_i \otimes \mathbf{e}_\alpha\,; \quad F_{i\alpha} = y_{i,\alpha}, \tag{11}$$

where $(\cdot)_{,\alpha} \equiv \partial(\cdot)/\partial x_\alpha$. One can now define the surface-analog of the Cauchy–Green strain of conventional continuum mechanics:

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = C_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta\,; \quad C_{\alpha\beta} = F_{i\alpha}F_{i\beta}. \tag{12}$$

As in the conventional theory, $\mathbf{C}$ is symmetric and non-negative definite, and thus may be represented in the form

$$\mathbf{C} = \lambda_1^2\mathbf{u}_1 \otimes \mathbf{u}_1 + \lambda_2^2\mathbf{u}_2 \otimes \mathbf{u}_2, \tag{13}$$

where λ_1, λ_2 ($\geqslant 0$) are the *principal stretches* and $\mathbf{u}_1$, $\mathbf{u}_2$ are the orthonormal principal vectors of strain.

To describe the mechanical behavior of the elastic surface, we assume the existence of a strain energy W, per unit area of Ω, that responds only to changes in the local intrinsic or metric geometry of the surface induced by the deformation. This formalizes the intuitive notion of a perfectly flexible surface. Thus, we assume that

$$W = \hat{W}(\mathbf{C}). \tag{14}$$

We note that W is automatically invariant under rigid body rotations. Non-uniform elastic properties may be taken into account by allowing $\hat{W}$ to depend explicitly on $\mathbf{x}$.

The formal proof of theorems in subsequent sections is facilitated by using the function of $\mathbf{F}$ defined by

$$W(\mathbf{F}) \equiv \hat{W}(\mathbf{F}^{\mathrm{T}}\mathbf{F}). \tag{15}$$

The surface-analog of the 2nd Piola–Kirchhoff stress is (Haseganu and Steigmann, 1994a):

$$\mathbf{S} = S_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta\,; \quad S_{\alpha\beta} = S_{\beta\alpha} = \partial\hat{W}/\partial C_{\alpha\beta} + \partial\hat{W}/\partial C_{\beta\alpha}, \tag{16}$$

and the analog of the Piola stress is

$$\mathbf{T} = \mathbf{F}\mathbf{S} = T_{i\alpha}\mathbf{e}_i \otimes \mathbf{e}_\alpha\,; \quad T_{i\alpha} = F_{i\beta}S_{\beta\alpha} = \partial W/\partial F_{i\alpha}. \tag{17}$$

This furnishes the force per unit reference length, $\boldsymbol{\tau}$, exerted by the material to the right of an embedded curve on the material to the left, according to the formula

$$\boldsymbol{\tau} = \mathbf{T}\boldsymbol{\nu}, \tag{18}$$

where $\boldsymbol{\nu} = \mathbf{x}'(s) \times \mathbf{e}_3$ is the *rightward* unit normal to the curve and $\mathbf{x}(s)$ is the arclength parametrization of the curve on Ω. If the surface is in equilibrium under applied or reactive edge forces, and the distributed forces (e.g. weight, pressure) are negligible, then the Piola stress satisfies

$$\operatorname{div} \mathbf{T} = \mathbf{0}\,; \quad T_{i\alpha,\alpha} = 0 \quad \text{in } \Omega. \tag{19}$$

In this work we study isotropic elastic surfaces. For these the strain energy is expressible as a symmetric function of the principal stretches (Naghdi and Tang, 1977; Haseganu and Steigmann, 1994a):

$$W(\mathbf{F}) = \hat{W}(\mathbf{C}) = w(\lambda_1, \lambda_2) = w(\lambda_2, \lambda_1). \tag{20}$$

The 2nd Piola–Kirchhoff stress is then given by

$$\mathbf{S} = \lambda_1^{-1} w_1 \mathbf{u}_1 \otimes \mathbf{u}_1 + \lambda_2^{-1} w_2 \mathbf{u}_2 \otimes \mathbf{u}_2, \tag{21}$$

where

$$w_\alpha = \partial w / \partial \lambda_\alpha, \tag{22}$$

and the associated Piola stress may be obtained from eqn (17).

3. VARIATIONAL THEORY

In this work we formulate conservative boundary value problems as minimization problems for appropriately defined potential energies. The so-called *relaxation* of the associated variational theory furnishes a rationale for the description and analysis of wrinkling in elastic surfaces and curves (Pipkin, 1986; Atai and Steigmann, 1997). Moreover, minimum energy states are relevant to the study of stable equilibria [e.g. Knops and Wilkes (1973); Como and Grimaldi (1995)].

We do not address the question of the existence of minimizers. Such matters have been studied extensively elsewhere (Ball, 1977; Dacarogna, 1989). Rather, we suppose that a given configuration minimizes a potential energy functional to be defined, and examine restrictions on the configuration imposed by certain well known necessary conditions in the calculus of variations. Chief among these are the Euler equations and certain inequalities generated by the *quasiconvexity* condition (Dacarogna, 1989). We then introduce a *relaxed* version of the theory which is quasiconvex in all configurations. The physical interpretation of the relaxed theory in terms of wrinkling or slackening is indicated briefly. Such interpretations have been thoroughly discussed in (Pipkin, 1986; Haseganu and Steigmann, 1994a; Atai and Steigmann, 1997). In some cases the relaxed theory admits a dual variational formulation leading to a minimum complementary energy principle and a uniqueness theorem for the equilibrium stress distribution.

These issues are addressed here in the context of the potential energy functional

$$E[\mathbf{y}] = \int_\Omega W(\mathbf{F})\, \mathrm{d}a + U[\mathbf{y}], \tag{23}$$

in which the integral represents the strain energy of the deformed elastic surface and the functional U represents the energetic contribution of one or more elastic curves (cables) interacting with the surface. This framework encompasses a wide variety of applications involving coupled one- and two-dimensional elastic elements [e.g. Steigmann and Li (1995); Steigmann and Ogden (1997)]. The often striking features of such interactions have recently been illustrated by Libai and Simmonds (1997).

In the applications considered here, the elastic cable is attached to a subset $P \subset \partial\Omega$ of the boundary of the membrane consisting of one or more arcs. We consider two types of attachment: (1) the cable is fixed to the membrane at each of its points and deforms with it as an embedded (material) curve; and (2) the endpoints of P are fixed and the configurations of the cable and the boundary of the membrane are congruent, but the interior points of the cable do not maintain fixed correspondences with points of the membrane. Thus, the cable and membrane may slide relative to each other without separating. In the context of tension structure design, the first alternative may be used to describe the stiffening effect of a seam-line along which the membrane is folded to provide a finished edge or to prevent fraying. The second alternative furnishes an idealized model of cables that slide freely through hoops stationed at intervals along the membrane boundary. In this application the stress in the membrane is controlled, at least to some degree, by the tensile force in the cable.

The functionals $U[\mathbf{y}]$ associated with the two types of attachment are

$$U_1[\mathbf{y}] = \int_P B(\mathbf{y}'(s))\,\mathrm{d}s \tag{24}$$

in the first instance, where $B(\cdot)$ is the cable strain energy function defined in Section 2, and

$$U_2[\mathbf{y}] = G(l[\mathbf{y}]) \equiv \int_L^{l[\mathbf{y}]} f(x/L)\,\mathrm{d}x \tag{25}$$

in the second instance, where

$$l[\mathbf{y}] = \int_P g(\mathbf{y}'(s))\,\mathrm{d}s; \quad g(\mathbf{y}') \equiv |\mathbf{y}'| \tag{26a,b}$$

is the total arclength of the image of P in a configuration $\mathbf{y}(P)$, $f(\cdot)$ is the force-extension relation of the elastic cable, and L is the arclength of P in the reference configuration. Here, $P \subset \partial\Omega$ is taken to consist of a single connected arc.

The functional U_2 represents the strain energy stored in a *homogeneously* strained cable. This form is used because the freely sliding cable is not subject to any tangential force distribution along its length. It then follows from eqns (7) and (8) that an elastically uniform cable is homogeneously strained in equilibrium configurations. The functional (25) is well defined for configurations that are not in equilibrium, but it is then to be interpreted as a potential rather than the total cable strain energy. In this respect U_2 is similar to the potentials associated with pressure acting at the boundary of a three-dimensional body (Fisher, 1988; Podio-Guidugli, 1988), or over the domain of a two-dimensional body (Steigmann, 1991; Bufler and Schneider, 1994). The *distribution* of pressure used in the definition of the potential is typically chosen to an *equilibrium* distribution for the fluid medium that transmits the pressure to the body in question. The same potential is then used to define a total potential energy functional for *all* kinematically admissible configurations, not just for those that are statically admissible as well. Apparently this subtle point has not been emphasized in the literature on configuration-dependent conservative loading. Functionals of this kind nevertheless furnish potentials for the problems considered.

3.1. *Equilibrium*

We obtain equilibrium equations for the coupled response of the elastic surface and curve from the stationarity of the energy $E[\mathbf{y}]$. Thus, let $\mathbf{y}(\mathbf{x};\varepsilon)$ be a one-parameter family of deformations with $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ for some $\varepsilon_0 > 0$. If the configuration corresponding to $\varepsilon = 0$ is equilibrated, then the Gateaux derivative of $E[\mathbf{y}(\mathbf{x};\varepsilon)]$ with respect to ε, evaluated at $\varepsilon = 0$, vanishes:

$$\int_\Omega \mathbf{T}\cdot \operatorname{grad}\mathbf{u}\,\mathrm{d}a + \dot{U} = 0. \tag{27}$$

Here the superposed dot indicates the value of the derivative at $\varepsilon = 0$, $\mathbf{T}(\mathbf{x})$ is the equilibrium Piola stress distribution, $\mathbf{u}(\mathbf{x}) = \dot{\mathbf{y}}$ is the *variation* of $\mathbf{y}$, grad $\mathbf{u}$ is its gradient, and the notation $\mathbf{A}\cdot\mathbf{B}$ is used to denote the scalar product, $A_{i\alpha}B_{i\alpha}$, of tensors $\mathbf{A}$, $\mathbf{B}$.

For the case $U = U_1$ it follows from eqns (1), (5) and (24) that

$$\dot{U} = \int_P \mathbf{f}\cdot\mathbf{u}'\,\mathrm{d}s, \tag{28}$$

where $\mathbf{f} = f(\lambda)\mathbf{t}$ is the curve–force (cf. eqn (7)) and $\mathbf{u}(s) = \mathbf{u}(\mathbf{x}(s))$ is the variation of $\mathbf{y}(\mathbf{x})$ evaluated on P. In the course of obtaining eqn (28), we used eqn (6) to deduce that $\dot{\lambda} = \lambda^{-1}\mathbf{r}'(s)\cdot\mathbf{u}'(s)$, together with the fact that $\mathbf{r}(s) = \mathbf{y}(\mathbf{x}(s))$ on P for the class of cable attachment under consideration.

Application of Green's theorem to eqns (27) and (28) results in

$$\int_{\partial\Omega\backslash P} \mathbf{T}\boldsymbol{\nu}\cdot\mathbf{u}\,\mathrm{d}s + \sum[\mathbf{f}\cdot\mathbf{u}]_{\partial P} + \int_P (\mathbf{T}\boldsymbol{\nu}-\mathbf{f}')\cdot\mathbf{u}\,\mathrm{d}s - \int_\Omega \mathbf{u}\cdot\operatorname{div}\mathbf{T}\,\mathrm{d}a = 0, \tag{29}$$

where the notation $[\cdot]_{\partial P}$ is used to denote the difference of the enclosed quantity at the endpoints of an arc of P and the sum extends over the individual arcs that comprise P. Regardless of the data assigned on $\partial\Omega\backslash P$, it follows immediately that E is stationary at $\varepsilon = 0$ only if eqn (19) is satisfied in Ω, only if

$$\mathbf{f} = \mathbf{0} \quad \text{on } \partial P_f, \tag{30}$$

where ∂P_f is an endpoint of P where position is not prescribed, and only if

$$\mathbf{T}\boldsymbol{\nu} = \mathbf{f}'(s) \quad \text{on } P. \tag{31}$$

The latter result also follows directly from eqn (8) on noting that the force per unit reference length, $\mathbf{b}(s)$, transmitted *to* the cable, is opposite to the traction, $\mathbf{T}\boldsymbol{\nu}$, exerted *by* the cable on the surface (Libai and Simmonds, 1997).

The remaining consequences of the stationary-energy principle follow from the vanishing of the first integral in eqn (29). Here we allow for the possibility, unusual in conventional elasticity, but commonplace in tension structure design, that the reference configuration Ω consists of disjoint subdomains. In practice, these sub-domains consist of pieces of fabric cut from a roll. These pieces, or *patterns*, are then fastened together, or *sutured*, prior to the erection of the assembled tension structure. In principle it is desirable to arrange the geometries of these patterns so as to achieve optimal conditions of stress or strain in the loaded structure. Often these optimality criteria are not stated precisely, if at all. For this and other reasons a definitive treatment of the patterning problem has yet to be achieved, despite the considerable effort that has been devoted to its solution. The work of Tabarrok and Qin (1992) is representative of current practice.

For illustrative purposes we suppose that Ω consists of just two regions Ω_1 and Ω_2. Let $S_1 \subset \partial\Omega_1$ and $S_2 \subset \partial\Omega_2$ be those parts of the boundaries of Ω_1 and Ω_2 that are sutured together. We suppose that $P \subset (\partial\Omega_1\backslash S_1) \cup (\partial\Omega_2\backslash S_2)$. On any part $\partial\Omega_t$ of $\partial\Omega_1\backslash S_1$ or $\partial\Omega_2\backslash S_2$ where position is not assigned, it follows from eqn (29) that the associated traction vanishes:

$$\mathbf{T}\boldsymbol{\nu} = \mathbf{0} \quad \text{on } \partial\Omega_t. \tag{32}$$

The remaining content of eqn (29) is then expressed by the requirement:

$$\int_{S_1} \mathbf{T}_1\boldsymbol{\nu}_1\cdot\mathbf{u}_1\,\mathrm{d}s + \int_{S_2} \mathbf{T}_2\boldsymbol{\nu}_2\cdot\mathbf{u}_2\,\mathrm{d}s = 0, \tag{33}$$

wherein the subscript 1 or 2 indicates evaluation on S_1 or S_2, respectively.

We now assume the existence of a diffeomorphism $S_2 = d(S_1)$ with local gradient $\alpha(s)$ defined on S_1. We further assume that the virtual displacement preserves the continuity of the sutured structure, so that $\mathbf{u}_2(d(s)) = \mathbf{u}_1(s) \equiv \mathbf{u}(s)$. Then eqn (33) reduces to

$$\int_{S_1} (\mathbf{T}_1 \mathbf{v}_1 + \alpha \mathbf{T}_2 \mathbf{v}_2) \cdot \mathbf{u} \, \mathrm{d}s = 0, \tag{34}$$

and we obtain

$$\mathbf{T}_1 \mathbf{v}_1 = -\alpha \mathbf{T}_2 \mathbf{v}_2 \quad \text{on } S_1. \tag{35}$$

As expected, this implies that for arbitrary arcs $s_1 \subset S_1$ and $s_2 = d(s_1) \subset S_2$, the net force transmitted by Ω_2 to Ω_1 along the suture is opposite to that transmitted by Ω_1 to Ω_2:

$$\int_{s_1} \mathbf{T}_1 \mathbf{v}_1 \, \mathrm{d}s = -\int_{s_2} \mathbf{T}_2 \mathbf{v}_2 \, \mathrm{d}s; \quad s_1 \subset S_1, s_2 = d(s_1). \tag{36}$$

The modifications to the foregoing argument required in the case of the freely sliding cable affect only the second and third terms in eqn (29). Equation (28) is now replaced by

$$\dot{U} = f(l/L)\dot{l}, \tag{37}$$

where

$$\dot{l} = \int_P \dot{g} \, \mathrm{d}s = \int_P \mathbf{t} \cdot \mathbf{u}' \, \mathrm{d}s, \tag{38}$$

$\mathbf{t}$ is the unit tangent to $\mathbf{y}(P)$, and we have made use of the variational form of eqn (26b):

$$\dot{g} = (\partial g/\partial y_i')u_i'; \quad \partial g/\partial y_i' = y_i'/|\mathbf{y}'| = t_i. \tag{39a,b}$$

Integrating eqn (38) by parts, recalling that ∂P is fixed for the present type of cable attachment and invoking the necessary conditions already derived, we find that eqn (29) reduces to

$$\int_P (\mathbf{T}\mathbf{v} - f\mathbf{t}') \cdot \mathbf{u} \, \mathrm{d}s = 0. \tag{40}$$

Thus, eqn (31) is replaced by

$$\mathbf{T}\mathbf{v} = f(l/L)\mathbf{t}'(s) \quad \text{on } P. \tag{41}$$

Since $\mathbf{t} \cdot \mathbf{t}' = 0$ it follows that the cable–membrane interaction is frictionless in the sense that the cable transmits no tangential traction to the membrane.

We remark that if the conditions on ∂P were relaxed to permit $\mathbf{u}$ to be non-zero at one of its points, then the stationarity of E would require that f vanish there, and hence everywhere in the cable. The alternative fixed-end conditions imposed here do not lead to this conclusion. Moreover, they correspond to the conditions that exist in actual tension structures, wherein cable pre-tension is controlled by adjusting the length of cable between the supports.

3.2. *Additional necessary conditions*

We obtain certain integral and algebraic inequalities that are satisfied by minimizers of the functional (23). Among these is the quasiconvexity condition that plays a fundamental role in Ball's existence theorems for nonlinear elasticity (Ball, 1977). Steigmann and Ogden (1997) recently extended Ball's proof of quasiconvexity to account for the presence of surface stress in plane-strain deformations of elastic solids. This extension generates restrictions analogous to quasiconvexity that apply to an elastic boundary, in addition to the condition previously known to apply to the bulk material. Apart from the restriction to two-dimensional deformations, the latter problem is similar to the type-1 cable–membrane interaction problem considered in the present work. For this reason we merely outline the main features of the derivation, with particular emphasis on the type-2 problem.

Thus suppose that $\mathbf{y}(\mathbf{x})$ is a minimizer of eqn (23) [with eqn (25)], and consider the perturbation

$$\mathbf{y}(\mathbf{x}) \to \mathbf{y}_\varepsilon(\mathbf{x}) = \mathbf{y}(\mathbf{x}) + \varepsilon\boldsymbol{\phi}(\mathbf{u})\,; \quad \mathbf{u} = \varepsilon^{-1}(\mathbf{x}-\mathbf{x}_0), \quad \varepsilon > 0, \tag{42}$$

where $\mathbf{x}_0 \in \Omega \cup P$ and $\boldsymbol{\phi}(\cdot)$ is a piecewise C^1 vector valued function, compactly supported in a region D containing the point $\mathbf{u} = \mathbf{0}$. ($\mathbf{u}$ should not be confused with the variations discussed previously.) The gradient of $\mathbf{y}_\varepsilon$ is

$$\mathbf{F}_\varepsilon(\mathbf{x}) = \mathbf{F}(\mathbf{x}) + \nabla\boldsymbol{\phi}(\mathbf{u}), \tag{43a}$$

where

$$\nabla\boldsymbol{\phi}(\mathbf{u}) = \frac{\partial \phi_i}{\partial u_\alpha}\mathbf{e}_i \otimes \mathbf{e}_\alpha \tag{43b}$$

is the gradient with respect to $\mathbf{u}$.

Since $\mathbf{y}(\mathbf{x})$ minimizes, we have

$$\int_{\Omega'} W(\mathbf{F}_\varepsilon(\mathbf{x}))\,\mathrm{d}a_x + G(l_\varepsilon) \geqslant \int_{\Omega'} W(\mathbf{F}(\mathbf{x}))\,\mathrm{d}a_x + G(l), \tag{44}$$

where $\mathrm{d}a_x$ is the area measure based on the variable $\mathbf{x}$, $\Omega' = \Omega \cap (x_0 + \varepsilon D)$, and $x_0 + \varepsilon D$ is the support of $\boldsymbol{\phi}(\mathbf{u}(\mathbf{x}))$. Here $G(\cdot)$ is the function defined in eqn (25). Its argument l_ε on the left-hand side of eqn (44) is given by eqn (26) with $\mathbf{y}_\varepsilon$ substituted in place of $\mathbf{y}$.

An estimate of the cable contribution to eqn (44) may be deduced from the expression

$$l_\varepsilon - l = \int_{P'} [g(\mathbf{y}'_\varepsilon(s_x)) - g(\mathbf{y}'(s_x))]\,\mathrm{d}s_x, \tag{45}$$

where $P' = P \cap (x_0 + \varepsilon D)$ and $\mathrm{d}s_x$ is the arclength measure associated with the variable $\mathbf{x}$. The change of variable $\mathbf{x} \to \mathbf{u}$ induces a transformation $s_x \to s_u$ with $\mathrm{d}s_x = \varepsilon\,\mathrm{d}s_u$. Then $s_x = s_x^0 + \varepsilon s_u$, where s_x^0 is the value of s_x at the point $\mathbf{x}_0 \in P$ corresponding to the point $\mathbf{u} = \mathbf{0}(s_u = 0)$ on P'. With the aid of eqn (42), eqn (45) may then be written

$$l_\varepsilon - l = \varepsilon \int_{P''} [g(\mathbf{y}'(s_x^0 + \varepsilon s_u) + \boldsymbol{\phi}'(s_u)) - g(\mathbf{y}'(s_x^0 + \varepsilon s_u))]\,\mathrm{d}s_u, \tag{46}$$

where $P'' = \mathbf{u}(P')$ is the image of P' under the mapping $\mathbf{x} \to \mathbf{u}$ and $\boldsymbol{\phi}'(s_u)$ is the tangential derivative of $\boldsymbol{\phi}(\mathbf{u}(s_u))$ on P''. With the support D of the function $\boldsymbol{\phi}(\mathbf{u})$ fixed, we obtain $l_\varepsilon - l = O(\varepsilon)$ as $\varepsilon \to 0^+$, and eqn (25) then furnishes the estimate

$$G(l_\varepsilon) - G(l) = \int_l^{l_\varepsilon} f(x/L)\,\mathrm{d}x = (l_\varepsilon - l)f(l/L) + o(\varepsilon), \tag{47}$$

where $f(l/L)$ is the force carried by the cable in the energy-minimizing configuration. Clearly this estimate applies only when $\mathbf{x}_0 \in P$: If $\mathbf{x}_0 \in \Omega \backslash P$, then P', P'' are empty for sufficiently small ε, and we obtain $G(l_\varepsilon) = G(l)$.

A similar estimate may be obtained for the remaining terms in eqn (44). We derive

$$\int_{\Omega'} [W(\mathbf{F}_\varepsilon(\mathbf{x})) - W(\mathbf{F}(\mathbf{x}))]\,\mathrm{d}a_x = \varepsilon^2 \int_{\Omega''} [W(\mathbf{F}(\mathbf{x}_0+\varepsilon\mathbf{u}) + \nabla\boldsymbol{\phi}(\mathbf{u})) - W(\mathbf{F}(\mathbf{x}_0+\varepsilon\mathbf{u}))]\,\mathrm{d}a_u, \tag{48}$$

where $\mathrm{d}a_u$ is the area measure based on the variable $\mathbf{u}$, and $\Omega'' = \mathbf{u}(\Omega')$ is the image of Ω' under the change of variable. The right-hand side is of order $O(\varepsilon^2)$ as $\varepsilon \to 0^+$, for any $\mathbf{x}_0 \in \Omega \cup P$.

For $\mathbf{x}_0 \in \Omega \backslash P$ we divide eqn (44) by ε^2 and let $\varepsilon \to 0^+$ to obtain the usual quasiconvexity inequality (Ball, 1977):

$$\int_D W(\mathbf{F}_0 + \nabla\boldsymbol{\phi}(\mathbf{u}))\,\mathrm{d}a_u \geqslant W(\mathbf{F}_0) \times (\text{area of } D), \tag{49}$$

where $\mathbf{F}_0 = \mathbf{F}(\mathbf{x}_0)$ and $\boldsymbol{\phi} = \mathbf{0}$ on ∂D. Necessary for this is the well known rank-one convexity condition (Ball, 1977). In the present context this takes the form

$$W(\mathbf{F}_0 + \mathbf{a} \otimes \mathbf{b}) - W(\mathbf{F}_0) \geqslant \mathbf{T}(\mathbf{F}_0) \cdot \mathbf{a} \otimes \mathbf{b}, \tag{50}$$

for all $\mathbf{a} = a_i\mathbf{e}_i$ and $\mathbf{b} = b_\alpha \mathbf{e}_\alpha$. In the limit of small $|\mathbf{a}|$ with $|\mathbf{b}|$ fixed, we obtain the Legendre–Hadamard inequality:

$$C^0_{i\alpha j\beta} a_i b_\alpha a_j b_\beta \geqslant 0, \tag{51}$$

where

$$C_{i\alpha j\beta}(\mathbf{F}) = \partial^2 W / \partial F_{i\alpha} \partial F_{j\beta} \tag{52}$$

and $C^0_{i\alpha j\beta} = C_{i\alpha j\beta}(\mathbf{F}_0)$.

An interesting consequence of eqn (51) that has no counterpart in the Legendre–Hadamard inequality for conventional two- or three-dimensional elasticity is the non-negativity of the 2nd Piola–Kirchhoff stress $\mathbf{S}_0$ furnished by $\mathbf{F}_0$ (Steigmann, 1986):

$$\mathbf{b} \cdot \mathbf{S}_0 \mathbf{b} \geqslant 0, \quad \forall \mathbf{b} = b_\alpha \mathbf{e}_\alpha. \tag{53}$$

For isotropic membranes, this is equivalent to [cf. eqn (22)]:

$$w_\alpha \geqslant 0; \quad \alpha = 1, 2. \tag{54}$$

A *sufficient* condition for quasiconvexity is ordinary convexity (Ball, 1977):

$$W(\mathbf{F}_0 + \operatorname{grad} \mathbf{v}(\mathbf{x})) - W(\mathbf{F}_0) \geqslant \mathbf{T}(\mathbf{F}_0) \cdot \operatorname{grad} \mathbf{v}(\mathbf{x}), \tag{55}$$

for all smooth $\mathbf{v}(\mathbf{x})$ that vanish on ∂D.

Independent necessary conditions for eqn (44) involving the cable alone may be obtained by choosing $\mathbf{x}_0 \in P$. In this case we divide eqn (44) by ε, invoke eqns (47) and (48), and pass to the limit $\varepsilon \to 0^+$ to derive

$$f(l/L) \int_{P''} [g(\mathbf{y}'_0 + \boldsymbol{\phi}'(s)) - g(\mathbf{y}'_0)]\,\mathrm{d}s \geqslant 0, \tag{56}$$

where $\mathbf{y}'_0$ is the value of $\mathbf{y}'(s_x)$ at s^0_x and $\boldsymbol{\phi} = \mathbf{0}$ on $\partial P''$. Necessary for this is the algebraic Weierstrass inequality (Bliss, 1946):

$$f(l/L)[g(\mathbf{y}_0'+\mathbf{a})-g(\mathbf{y}_0')-a_i(\partial g/\partial y_i')_0] \geqslant 0, \tag{57}$$

for arbitrary $\mathbf{a} = a_i\mathbf{e}_i$. The notation $(\cdot)_0$ in the last term indicates evaluation of the enclosed derivative at $\mathbf{y}_0'$. Conversely, integration of eqn (57) with $\mathbf{a} = \boldsymbol{\phi}'(s)$ yields eqn (56) for all $\boldsymbol{\phi}$ that vanish on $\partial P''$. Thus, eqns (56) and (57) are equivalent. The latter inequality may be simplified by substituting eqn (39b) to obtain

$$f(l/L)(|\mathbf{y}_0'+\mathbf{a}|-|\mathbf{y}_0'|-\mathbf{a}\cdot\mathbf{t}_0) \geqslant 0, \quad \forall\mathbf{a}, \tag{58}$$

where $\mathbf{t}_0$ is the value of $\mathbf{t}$ at the point s_x^0 on P. We decompose $\mathbf{a}$ in the form

$$\mathbf{a} = (\mathbf{a}\cdot\mathbf{t}_0)\mathbf{t}_0+\mathbf{a}_\perp, \quad \mathbf{t}_0\cdot\mathbf{a}_\perp = 0. \tag{59}$$

Hence

$$|\mathbf{y}_0'+\mathbf{a}| = [(|\mathbf{y}_0'|+\mathbf{a}\cdot\mathbf{t}_0)^2+|\mathbf{a}_\perp|^2]^{1/2} \geqslant |\mathbf{y}_0'|+\mathbf{a}\cdot\mathbf{t}_0, \tag{60}$$

with equality if and only if $\mathbf{a}_\perp = \mathbf{0}$ and $\mathbf{a}\cdot\mathbf{t}_0 \geqslant -|\mathbf{y}_0'|$. Thus, eqn (58) is equivalent to the requirement that the cable force be tensile in a minimizing configuration:

$$f(l/L) \geqslant 0. \tag{61}$$

We note that application of the triangle inequality to eqn (58) yields no information.

For the type-1 edge reinforcement problem, the development that led to eqn (56) now yields

$$\int_{P''} [B(\mathbf{y}_0'+\boldsymbol{\phi}'(s))-B(\mathbf{y}_0')]\,\mathrm{d}s \geqslant 0, \tag{62}$$

and the associated Weierstrass condition is

$$B(\mathbf{y}_0'+\mathbf{a})-B(\mathbf{y}_0') \geqslant \mathbf{f}_0\cdot\mathbf{a}, \quad \forall\mathbf{a}, \tag{63}$$

where $\mathbf{f}_0 = \hat{\mathbf{f}}(\mathbf{y}_0')$ (cf. eqn (3)). This is equivalent to the *two* inequalities (Atai and Steigmann, 1997):

$$f(\lambda) \geqslant 0, \quad \hat{B}(\mu)-\hat{B}(\lambda) \geqslant (\mu-\lambda)f(\lambda), \quad \forall\mu > 0, \tag{64a,b}$$

where $\lambda = |\mathbf{y}_0'|$ is the local curve stretch in a minimizing configuration. Thus, the cable is under tension and the strain energy $\hat{B}(\cdot)$ is convex at λ. The latter result requires that the tangent modulus be non-negative: $f'(\lambda) \geqslant 0$.

3.3. *Relaxed theory*

Strain energy functions typically used in the theories of elastic curves and surfaces do not satisfy the necessary conditions (64a) or (53), respectively, at all values of the strain. Thus, energy-minimizing configurations generally do not exist for a large class of boundary value problems. Two main approaches to restoring well-posedness have been discussed in the literature. The first, known as *regularization*, is to replace the model by one having additional structure, in which restrictions such as eqns (64a) or (53) do not arise. In the present context, regularization may be achieved by substituting shell and rod models in place of the simpler membrane and cable theories. The alternative to regularization, known as *relaxation*, is to modify the constitutive equations of the membrane and cable so that restrictions like eqns (53) and (64) are automatically satisfied. The task is then to relate the relaxed model to the original model in a way that is physically meaningful.

For example, the relaxed minimization problem for the type-1 cable-membrane model is based on the modified energy functional (Dacarogna, 1989):

$$E_R[\mathbf{y}] = \int_\Omega W_R(\mathbf{F})\,\mathrm{d}a + \int_P B_R(\mathbf{y}')\,\mathrm{d}s, \tag{65}$$

where $W_R(\mathbf{F})$ is the *quasiconvexification* of $W(\mathbf{F})$:

$$W_R(\mathbf{F}) = \overset{\sup}{\phi}\ \{\phi : \phi(\mathbf{F}) \leqslant W(\mathbf{F}), \quad \text{and } \phi \text{ quasiconvex}\}, \tag{66}$$

and $B_R(\mathbf{y}')$ is the *convexification* of $B(\mathbf{y}')$:

$$B_R(\mathbf{y}') = \overset{\sup}{\psi}\ \{\psi : \psi(\mathbf{y}') \leqslant B(\mathbf{y}'), \quad \text{and } \psi \text{ convex}\}. \tag{67}$$

Granted suitable bounds on these functions, it is possible to show that the minimization problem for E_R has a solution in an appropriate function space, and that

$$\min E_R[\mathbf{y}] = \inf E[\mathbf{y}], \tag{68}$$

even when $E[\mathbf{y}]$ fails to have a minimizer. In particular, if $\mathbf{y}(\mathbf{x})$ minimizes E_R then it is typically possible to construct a minimizing sequence $\{\mathbf{y}_n\}$ for E, converging (weakly) to $\mathbf{y}(\mathbf{x})$, for which $E[\mathbf{y}_n] \to E_R[\mathbf{y}]$. We refer to Acerbi and Fusco (1984) and Dacarogna (1989) for detailed discussions of these concepts.

If the strain energies $W(\mathbf{F})$ and $B(\mathbf{y}')$ are unequal to their relaxations, then minimizing sequences typically exhibit finer and finer scale discontinuities in the gradients of the $\mathbf{y}_n(\mathbf{x})$ as n increases. In the limit the sequence itself converges to a smooth function, but the limit of the sequence of gradients is discontinuous everywhere. In the context of membrane and cable theories, this fine scale structure may be interpreted in terms of a continuous distribution of wrinkles of infinitesimal amplitude, spaced an infinitesimal distance apart (Pipkin, 1986; Atai and Steigmann, 1997). In a real membrane or cable small, but finite, flexural stiffness intervenes to prevent the attainment of an infinitely fine distribution. It is in this sense that the relaxed version of the theory entails a degree of idealization. Despite this limitation, such an approach is far more tractable than regularization, and furnishes a useful model for calculating global features of the response when the original model fails to possess a solution.

We remark that minimizing sequences involving wrinkling have been constructed for membranes and cables separately, but not for the membrane–cable combinations considered here. Thus, the attainability of the relaxed energy from the original energy remains an open question in the present context.

3.3.1. *Relaxed membrane energy*. Kohn and Strang (1986) have observed that W_R is bounded above and below by the *rank-one convexification* and the *convexification* of W, respectively. These are defined as in eqn (66), except that ϕ is rank-one convex or convex as appropriate. The latter functions are defined by algebraic inequalities rather than the integral inequality that defines quasiconvexity. Thus, they are generally easier to compute than W_R.

In membrane theory there are many examples in which the two types of convexification are not only easy to compute, but actually coincide. In such cases W_R is obtained directly (Pipkin, 1986). The relaxed strain energy is then convex as a function of $\mathbf{F}$, i.e.

$$W_R(\mathbf{F}+\Delta\mathbf{F}) - W_R(\mathbf{F}) \geqslant \hat{\mathbf{T}}(\mathbf{F}) \cdot \Delta\mathbf{F}, \tag{69}$$

where $\hat{\mathbf{T}}(\cdot)$ is the constitutive equation derived from W_R:

$$\hat{\mathbf{T}}(\mathbf{F}) = \hat{T}_{i\alpha}(\mathbf{F})\mathbf{e}_i \otimes \mathbf{e}_\alpha; \quad \hat{T}_{i\alpha}(\mathbf{F}) = \partial W_{\mathrm{R}}/\partial F_{i\alpha}. \tag{70}$$

This property leads to global minimum principles for the potential energy functional and a complementary energy functional to be specified in Section 4 below.

Now any rank-one convex function necessarily satisfies the Legendre–Hadamard inequality (51), and any convex function satisfies the condition of local convexity obtained by substituting arbitrary $A_{i\alpha}$ in place of $a_i b_\alpha$ in eqn (51). Pipkin (1986) has derived inequalities for isotropic elastic membranes that are equivalent to the two types of local convexity. In particular, the Legendre–Hadamard inequality is equivalent to inequalities (54) and

$$w_{11} \geqslant 0, \quad w_{22} \geqslant 0, \quad a \geqslant 0, \tag{71a}$$

$$(w_{11}w_{22})^{1/2} - w_{12} \geqslant b - a, \quad (w_{11}w_{22})^{1/2} + w_{12} \geqslant -b - a, \tag{71b}$$

where

$$a = (\lambda_1 w_1 - \lambda_2 w_2)/(\lambda_1^2 - \lambda_2^2), \quad b = (\lambda_2 w_1 - \lambda_1 w_2)/(\lambda_1^2 - \lambda_2^2) \tag{72}$$

with $w_\alpha = \partial w/\partial \lambda_\alpha$ and $w_{\alpha\beta} = \partial^2 w/\partial \lambda_\alpha \partial \lambda_\beta$. For $\lambda_1 = \lambda_2$, the results obtained by applying L'Hôpital's rule to eqns (71), (72) remain valid. Alternatively, the local convexity inequality is equivalent to eqns (54) and (71a), together with

$$w_{11}w_{22} - w_{12}^2 \geqslant 0, \quad |a| \geqslant b \tag{73}$$

in place of eqn (71b) (Pipkin, 1986).

It is frequently the case that inequalities (71) and (73) are satisfied by a given strain energy function $w(\lambda_1, \lambda_2)$. The potential violation of eqn (54) is then the only cause of the failure of quasiconvexity. In such circumstances the relaxed strain energy, expressed as a function of the stretches, is the symmetric composite function defined by (Pipkin, 1986):

$$\begin{aligned} w_{\mathrm{R}}(\lambda_1, \lambda_2) = {} & w(\lambda_1, \lambda_2); \quad \lambda_1 \geqslant v(\lambda_2), \quad \lambda_2 \geqslant v(\lambda_1) \\ & \hat{w}(\lambda_1); \quad \lambda_1 > 1, \quad \lambda_2 \leqslant v(\lambda_1) \\ & \hat{w}(\lambda_2); \quad \lambda_2 > 1, \quad \lambda_1 \leqslant v(\lambda_2) \\ & 0; \quad \lambda_1 \leqslant 1, \quad \lambda_2 \leqslant 1 \end{aligned} \tag{74}$$

provided that $w(1,1) = w_\alpha(1,1) = 0$. Here

$$\hat{w}(x) = w(x, v(x)) = w(v(x), x) \tag{75}$$

and $v(x)$ is a solution of

$$w_2(x, v(x)) = w_1(v(x), x) = 0. \tag{76}$$

We assume this solution to be unique.

Physically, $v(x)$ is the transverse stretch that nullifies the transverse stress when the longitudinal stretch assumes some value $x > 1$ in uniaxial tension. Further reduction of the transverse stretch at the same value of x is accomplished by fine scale wrinkling perpendicular to the tensile axis. Wrinkling does not alter the strain energy, which retains the value $\hat{w}(x)$ associated with uniaxial tension. This is the interpretation of the second and third branches of eqn (74). The fourth branch corresponds to slack states generated by simultaneous wrinkling along two principal directions. We refer to (Pipkin, 1986) for detailed explanations of these ideas.

The relaxed energy defined by eqn (74) is locally convex as a function of $\mathbf{F}$ for all λ_1, $\lambda_2 \geqslant 0$, and thus in all of $\mathbf{F}$-space. Since the latter region is convex, it follows that the global convexity inequality (69) is also satisfied, as discussed previously. (We remark that the unilateral constraint on the determinant of the deformation gradient in conventional elasticity has no counterpart here as the determinant of **F** in eqn (11) is not defined.)

In this work we use several membrane strain energy functions whose relaxations meet the foregoing conditions. Each of these involves a material constant μ with dimensions of force/length (or energy/area). This constant may be interpreted as the bulk shear modulus of the material at infinitesimal strain, multiplied by the uniform initial thickness of the membrane. For example, the relaxation of the neo-Hookean strain energy is defined by eqns (74) and (75) with (Pipkin, 1986):

$$w(\lambda_1,\lambda_2) = \tfrac{1}{2}\mu(\lambda_1^2+\lambda_2^2+\lambda_1^{-2}\lambda_2^{-2}-3), \quad \hat{w}(x) = \tfrac{1}{2}\mu(x^2+2x^{-1}-3), \quad v(x) = x^{-1/2}, \tag{77}$$

and the relaxed form of the Varga strain energy (Varga, 1966) is given by (Haseganu and Steigmann, 1994b):

$$w(\lambda_1,\lambda_2) = 2\mu(\lambda_1+\lambda_2+\lambda_1^{-1}\lambda_2^{-1}-3), \quad \hat{w}(x) = 2\mu(x+2x^{-1/2}-3), \quad v(x) = x^{-1/2}. \tag{78}$$

We also study harmonic materials (Varley and Cumberbatch, 1980). For these the (unrelaxed) strain energy is of the form

$$w(\lambda_1,\lambda_2) = 2\mu[F(I)-J]; \quad I = \lambda_1+\lambda_2, \quad J = \lambda_1\lambda_2 \tag{79}$$

for some function $F(\cdot)$. In particular, we consider the special case of the *standard linear solid* (Wu, 1979):

$$F(I) = \frac{\bar{\lambda}+2\mu}{4\mu}I^2 + \frac{\bar{\lambda}+\mu}{\mu}(1-I), \tag{80}$$

where $\bar{\lambda}$ and μ are the usual Lamé moduli times the membrane thickness, the overbar being used so as not to confuse the modulus with the cable stretch. On the first branch of eqn (74) where $w_\alpha > 0$, it follows by adding the expressions for w_α obtained from eqn (79) that $F'(I) > I/2$. According to eqn (80) this corresponds to the region defined by $I > 2$ in the (λ_1, λ_2)-plane. Elsewhere the remaining branches of eqn (74) are used, with

$$\hat{w}(x) = 2\mu[F(x+v(x))-xv(x)], \quad v(x) = \frac{2(\bar{\lambda}+\mu)}{\bar{\lambda}+2\mu} - \frac{\bar{\lambda}}{\bar{\lambda}+2\mu}x. \tag{81}$$

We note that values of x for which $v(x) < 0$ lie outside the range in which eqn (80) furnishes reasonable agreement with data on real materials. If we exclude such values then it is straightforward to show that the relaxed energy defined by eqn (74) and eqns (79), (81) satisfies eqn (54) and the convexity conditions (71a) and (73) provided that

$$\mu > 0, \quad \bar{\lambda}+\mu > 0. \tag{82a,b}$$

3.3.2. *Relaxed cable energy*. The construction of the relaxed cable strain energy function is far simpler. We assume that $\hat{B}(\lambda) = B(\mathbf{y}')$ is a convex function of the stretch with an isolated minimum at $\lambda = 1$. Then eqn (64b) is satisfied for all positive λ and $f = \hat{B}'(\lambda)$ violates eqn (64a) if and only if $\lambda < 1$. The relaxed energy $B_R(\mathbf{y}')$ is the function of $\lambda = |\mathbf{y}'|$ defined by (Atai and Steigmann, 1997):

$$\hat{B}_R(\lambda) = \hat{B}(\lambda), \quad \lambda \geqslant 1$$
$$0, 0 \leqslant \lambda < 1. \tag{83}$$

Here, too, the second branch may be interpreted in terms of a continuous distribution of wrinkles. The associated constitutive relation is

$$\mathbf{f} = \hat{\mathbf{f}}_R(\mathbf{y}') = f_R(|\mathbf{y}'|)\mathbf{y}'/|\mathbf{y}'|, \tag{84}$$

where

$$f_R(\lambda) = \begin{cases} f(\lambda), & \lambda \geqslant 1 \\ 0, & 0 \leqslant \lambda < 1. \end{cases} \tag{85}$$

The function $B_R(\mathbf{y}')$ thus defined is convex, since it satisfies eqn (64a, b), and hence eqn (63), for all values of its argument.

In this work we use a simple strain energy that conforms to the foregoing requirements:

$$\hat{B}(\lambda) = \tfrac{1}{2}E(\lambda-1)^2; \quad f(\lambda) = E(\lambda-1), \tag{86a,b}$$

where E is a positive constant with dimensions of force.

For the type-2 cable-membrane problem, inequalities (64a, b) are replaced by the single inequality (61). Strictly speaking, this need only apply at stretches associated with minimizing configurations. Thus the *function* $G(\cdot)$ in eqn (25) need not be convex. More precisely, our methods do not yield convexity as a necessary condition in this case. However, it is natural to impose the requirement that $f(l/L)$ be a non-decreasing function, in accordance with eqn (64b). Thus we assume that $G(\cdot)$ is convex on $(0, \infty)$ with an isolated minimum at $l = L$, and define its relaxation by

$$G_R(l) = \begin{cases} G(l), & l \geqslant L \\ 0, & 0 \leqslant l < L, \end{cases} \tag{87}$$

where L is the unstretched length of the cable. The associated force–extension relation is $f_R(l/L) = G'_R(l)$. This is equivalent to eqn (85) with $f(\lambda)$ replaced by $G'(l)$, and the relaxed potential energy is now given by

$$E_R[\mathbf{y}] = \int_\Omega W_R(\mathbf{F})\,\mathrm{d}a + G_R(l[\mathbf{y}]). \tag{88}$$

The remainder of this work is based on the relaxed energies (65) and (88). Thus, we drop the subscript R in the subsequent development.

4. MINIMUM ENERGY AND MINIMUM COMPLEMENTARY ENERGY

4.1. *Legendre–Fenchel transformations*

The convexity of the relaxed variational problem may be used to establish a dual variational principle based on a complementary strain energy. Here we adapt the methods of Gao and Strang (1989), Gao (1992), and Pipkin (1994) to derive the dual formulation for the coupled cable–membrane problem. A similar formulation was developed for the theory of discrete elastic networks by Atai and Steigmann (1997) and for the membrane theory of elastic networks by Haseganu and Steigmann (1996).

Let $W(\mathbf{F})$ be the (relaxed) membrane strain energy function. For given Piola stress $\mathbf{T}$, the complementary energy, $W^*(\mathbf{T})$, is defined to be the Legendre–Fenchel transformation of $W(\cdot)$:

$$W^*(\mathbf{T}) = \sup_{\mathbf{F}} [\mathbf{T}\cdot\mathbf{F} - W(\mathbf{F})]. \tag{89}$$

Under appropriate growth conditions on $W(\cdot)$ (e.g. $W(\mathbf{F})/|\mathbf{F}| \to \infty$ as $|\mathbf{F}| \to \infty$), the bracketed expression is maximized by some (possibly non-unique) deformation gradient $\mathbf{F}_0$. Thus

$$W^*(\mathbf{T}) + W(\mathbf{F}_0) = \mathbf{T}\cdot\mathbf{F}_0\,; \quad W^*(\mathbf{T}) + W(\mathbf{F}) \geqslant \mathbf{T}\cdot\mathbf{F}, \quad \forall(\mathbf{T},\mathbf{F}). \tag{90}$$

Furthermore, the convexity of $W(\cdot)$ implies that $\mathbf{F}_0$ is a maximizer if and only if $\mathbf{T} = \hat{\mathbf{T}}(\mathbf{F})$, where $\hat{\mathbf{T}}(\cdot)$ is the constitutive relation (70) based on the relaxed strain energy. To prove this we adopt the mild constitutive restriction [obeyed by the functions (77)–(80)] that the original strain energy is a C^2 function of $\mathbf{F}$. The relaxed energy is then easily seen to be C^1 and piecewise C^2. Thus, if $\mathbf{F}_0$ is a maximizer, the bracketed expression in eqn (89) is stationary at $\mathbf{F}_0$, and $\mathbf{T} = \hat{\mathbf{T}}(\mathbf{F}_0)$. Conversely, if $\mathbf{T} = \hat{\mathbf{T}}(\mathbf{F}_0)$ then by the convexity of $W(\cdot)$,

$$W(\mathbf{F}) - W(\mathbf{F}_0) \geqslant (\mathbf{F}-\mathbf{F}_0)\cdot\hat{\mathbf{T}}(\mathbf{F}_0), \quad \text{or } \mathbf{T}\cdot\mathbf{F} - W(\mathbf{F}) \leqslant \mathbf{T}\cdot\mathbf{F}_0 - W(\mathbf{F}_0) \tag{91}$$

so $\mathbf{F}_0$ is a maximizer. Thus we have shown that

$$W^*(\mathbf{T}) + W(\mathbf{F}) \geqslant \mathbf{T}\cdot\mathbf{F}, \quad \forall(\mathbf{T},\mathbf{F}) \tag{92a}$$

and

$$W^*(\mathbf{T}) + W(\mathbf{F}) = \mathbf{T}\cdot\mathbf{F} \quad \text{if and only if } \mathbf{T} = \hat{\mathbf{T}}(\mathbf{F}). \tag{92b}$$

The latter equality does *not* imply that $\mathbf{F}$ is uniquely determined by $\mathbf{T}$.

Results like these are standard in convex analysis [e.g. van Tiel (1984)]. Pipkin (1994) was apparently the first to apply them to membrane theory. Gao and Strang (1989) and Gao (1992) have presented similar discussions of convex duality in the setting of conventional nonlinear elasticity.

A parallel development for elastic curves was given by Atai and Steigmann (1997). Thus, let $B(\mathbf{y}')$ be the (relaxed) energy per unit length of $P \subset \partial\Omega$. The associated complementary energy is

$$B^*(\mathbf{f}) = \sup_{\mathbf{y}'} [\mathbf{f}\cdot\mathbf{y}' - B(\mathbf{y}')], \tag{93}$$

where $\mathbf{f}$ is the cable force. The convexity of $B(\cdot)$ yields

$$B^*(\mathbf{f}) + B(\mathbf{y}') \geqslant \mathbf{f}\cdot\mathbf{y}', \quad \forall(\mathbf{f},\mathbf{y}') \tag{94a}$$

and

$$B^*(\mathbf{f}) + B(\mathbf{y}') = \mathbf{f}\cdot\mathbf{y}' \quad \text{if and only if } \mathbf{f} = \hat{\mathbf{f}}(\mathbf{y}'), \tag{94b}$$

where $\hat{\mathbf{f}}(\cdot)$ is the constitutive function (84) derived from the relaxed energy.

For the cable potential defined by eqn (87), the dual is

$$G^*(f) = \sup_{l} [fl - G(l)]. \tag{95}$$

Once again the convexity of $G(\cdot)$ yields

$$G^*(f) + G(l) \geqslant fl, \quad \forall(f,l) \tag{96a}$$

and

$$G^*(f)+G(l) = fl \quad \text{if and only if } f = \hat{f}(l), \tag{96b}$$

where $\hat{f}(l) = f_R(l/L)$.

4.2. *Complementary potential energy and uniqueness of stress*

Let $\tilde{\mathbf{T}}(\mathbf{x})$ be a statically admissible Piola stress field, so that $\operatorname{div}\tilde{\mathbf{T}} = \mathbf{0}$ in Ω, $\tilde{\mathbf{T}}\mathbf{v} = \mathbf{0}$ on $\partial\Omega_t$, and eqn (35) is satisfied if the surface is sutured. Let $\tilde{\mathbf{F}}(\mathbf{x})$ be the gradient of a kinematically admissible deformation $\tilde{\mathbf{y}}(\mathbf{x})$. Here we take kinematic admissibility to mean that $\tilde{\mathbf{y}}$ is continuous in Ω, continuously differentiable in any connected subregion of Ω, and satisfies $\tilde{\mathbf{y}}(\mathbf{x}) = \mathbf{y}_0(\mathbf{x})$ on $\partial\Omega_y$. Then by the virtual work theorem,

$$\int_\Omega \tilde{\mathbf{T}}\cdot\tilde{\mathbf{F}}\,\mathrm{d}a = \int_{\partial\Omega} \tilde{\mathbf{y}}\cdot\tilde{\mathbf{T}}\mathbf{v}\,\mathrm{d}s = \int_{\partial\Omega_y} \mathbf{y}_0\cdot\tilde{\mathbf{T}}\mathbf{v}\,\mathrm{d}s + \int_P \tilde{\mathbf{y}}\cdot\tilde{\mathbf{T}}\mathbf{v}\,\mathrm{d}s, \tag{97}$$

and eqns (92a, b) yield

$$\int_\Omega [W(\tilde{\mathbf{F}}) + W^*(\tilde{\mathbf{T}})]\,\mathrm{d}a \geqslant \int_{\partial\Omega_y} \mathbf{y}_0\cdot\tilde{\mathbf{T}}\mathbf{v}\,\mathrm{d}s + \int_P \tilde{\mathbf{y}}\cdot\tilde{\mathbf{T}}\mathbf{v}\,\mathrm{d}s, \tag{98}$$

with equality if and only if $\tilde{\mathbf{T}} = \hat{\mathbf{T}}(\tilde{\mathbf{F}})$ almost everywhere in Ω and on $\partial\Omega_y \cup P$.

For the type-1 cable membrane problem, suppose the cable force $\tilde{\mathbf{f}}(s)$ is statically admissible in the sense that eqns (30) and (31) are satisfied, i.e.

$$\tilde{\mathbf{T}}\mathbf{v} = \tilde{\mathbf{f}}'(s) \quad \text{on } P, \quad \tilde{\mathbf{f}} = \mathbf{0} \quad \text{on } \partial P_f. \tag{99}$$

We require that $\tilde{\mathbf{y}}(\mathbf{x}(s)) = \hat{\mathbf{y}}$ at points $\partial P_y = \partial P \backslash \partial P_f$ where position $\hat{\mathbf{y}}$ is assigned, and that $\hat{\mathbf{y}} = \mathbf{y}_0$ at any point $\mathbf{x}$ belonging to $\partial\Omega_y \cap \partial P_y$. Thus

$$\int_P \tilde{\mathbf{f}}\cdot\tilde{\mathbf{y}}'(s)\,\mathrm{d}s = \sum [\hat{\mathbf{y}}\cdot\tilde{\mathbf{f}}]_{\partial P_y} - \int_P \tilde{\mathbf{y}}\cdot\tilde{\mathbf{T}}\mathbf{v}\,\mathrm{d}s, \tag{100}$$

and eqn (94a) yields

$$\int_P [B(\tilde{\mathbf{y}}') + B^*(\tilde{\mathbf{f}})]\,\mathrm{d}s \geqslant \sum [\hat{\mathbf{y}}\cdot\tilde{\mathbf{f}}]_{\partial P_y} - \int_P \tilde{\mathbf{y}}\cdot\tilde{\mathbf{T}}\mathbf{v}\,\mathrm{d}s. \tag{101}$$

Again, equality applies if and only if $\tilde{\mathbf{f}} = \hat{\mathbf{f}}(\tilde{\mathbf{y}}')$.

For statically admissible $\tilde{\mathbf{T}}$ and $\tilde{\mathbf{f}}$, we define the complementary potential energy

$$E^*[\tilde{\mathbf{T}},\tilde{\mathbf{f}}] = \int_\Omega W^*(\tilde{\mathbf{T}})\,\mathrm{d}a + \int_P B^*(\tilde{\mathbf{f}})\,\mathrm{d}s - \int_{\partial\Omega_y} \mathbf{y}_0\cdot\tilde{\mathbf{T}}\mathbf{v}\,\mathrm{d}s - \sum [\hat{\mathbf{y}}\cdot\tilde{\mathbf{f}}]_{\partial P_y}. \tag{102}$$

The potential energy is given by eqn (23):

$$E[\tilde{\mathbf{y}}] = \int_\Omega W(\tilde{\mathbf{F}})\,\mathrm{d}a + \int_P B(\tilde{\mathbf{y}}')\,\mathrm{d}s, \tag{103}$$

and eqns (98) and (101) combine to give

$$E[\tilde{\mathbf{y}}] + E^*[\tilde{\mathbf{T}},\tilde{\mathbf{f}}] \geqslant 0, \quad \forall(\tilde{\mathbf{y}},\tilde{\mathbf{T}},\tilde{\mathbf{f}}) \tag{104a}$$

with

$$E[\mathbf{y}]+E^*[\mathbf{T},\mathbf{f}]=0 \tag{104b}$$

if and only if $\mathbf{T}(\mathbf{x})=\hat{\mathbf{T}}(\operatorname{grad}\mathbf{y}(\mathbf{x}))$ in Ω and $\mathbf{f}=\hat{\mathbf{f}}(\mathbf{y}')$ on P.

Now let $\mathbf{y}(\mathbf{x})$ be an *equilibrium* deformation. Then $\mathbf{T}(\mathbf{x})=\hat{\mathbf{T}}(\operatorname{grad}\mathbf{y}(\mathbf{x}))$ and $\mathbf{f}(s)=\tilde{\mathbf{f}}(\mathbf{y}'(s))$ are statically admissible, and eqns (104a, b) yield

$$E[\tilde{\mathbf{y}}]\geqslant -E^*[\mathbf{T},\mathbf{f}]=E[\mathbf{y}], \tag{105a}$$

so that $\mathbf{y}(\mathbf{x})$ minimizes $E[\cdot]$ absolutely among kinematically admissible alternatives. Similarly,

$$E^*[\tilde{\mathbf{T}},\tilde{\mathbf{f}}]\geqslant -E[\mathbf{y}]=E^*[\mathbf{T},\mathbf{f}]. \tag{105b}$$

Thus the stress $\mathbf{T}(\mathbf{x})$ and the cable force $\mathbf{f}(s)$ furnished by an equilibrium deformation are global minimizers of $E^*[\cdot]$.

We note that the common values of $E[\mathbf{y}]$ and $-E^*[\mathbf{T},\mathbf{f}]$ are bounded above and below by $E[\tilde{\mathbf{y}}]$ and $-E^*[\tilde{\mathbf{T}},\tilde{\mathbf{f}}]$, respectively. These bounds may be used to generate approximate solutions to equilibrium boundary value problems.

The modifications to the foregoing development required in the case of the type-2 problem are minor. Equation (99a) remains valid, but with $\tilde{\mathbf{f}}(s)$ replaced by $\tilde{\mathbf{f}}=\tilde{f}\tilde{\mathbf{t}}$, where $\tilde{f}$ is a non-negative constant and $\tilde{\mathbf{t}}$ is the unit tangent to $\tilde{\mathbf{y}}(P)$ at arclength station s. Equation (99b) is not applicable. Equation (100) also applies, but the left-hand side may now be simplified:

$$\int_P \tilde{\mathbf{f}}\cdot\tilde{\mathbf{y}}'\,\mathrm{d}s=\tilde{f}\tilde{l};\quad \tilde{l}=\int_P|\tilde{\mathbf{y}}'(s)|\,\mathrm{d}s. \tag{106}$$

On combining this with eqns (100) and (96a, b) we obtain

$$G^*(\tilde{f})+G(\tilde{l})\geqslant\tilde{f}\tilde{l}=[\hat{\mathbf{y}}\cdot\tilde{\mathbf{f}}]_{\partial P}-\int_P\tilde{\mathbf{y}}\cdot\tilde{\mathbf{T}}\mathbf{v}\,\mathrm{d}s, \tag{107}$$

with equality if, and only if, $\tilde{\mathbf{T}}(s)=\hat{\mathbf{T}}(\operatorname{grad}\tilde{\mathbf{y}}(\mathbf{x}(s))$ on P and $\tilde{f}=\hat{f}(\tilde{l})$, where $\hat{f}(x)=f_R(x/L)$.

This leads us to define the complementary energy

$$E^*[\tilde{\mathbf{T}},\tilde{f}]=\int_\Omega W^*(\tilde{\mathbf{T}})\,\mathrm{d}a+G^*(\tilde{f})-\int_{\partial\Omega_y}\mathbf{y}_0\cdot\tilde{\mathbf{T}}\mathbf{v}\,\mathrm{d}s-[\hat{\mathbf{y}}\cdot\tilde{\mathbf{f}}]_{\partial P}. \tag{108}$$

The potential energy is now given by

$$E[\tilde{\mathbf{y}}]=\int_\Omega W(\tilde{\mathbf{F}})\,\mathrm{d}a+G(\tilde{l}), \tag{109}$$

and it is straightforward to show that the obvious modifications to eqns (104) and (105) apply.

The foregoing minimum principles may be used to prove uniqueness of the equilibrium stress and cable force distributions. To see this we adapt Pipkin's analysis to the type-1 problem as follows (Pipkin, 1994): Let $\{\mathbf{T}_1(\mathbf{x}),\mathbf{f}_1(s),\mathbf{y}_1(\mathbf{x})\}$ and $\{\mathbf{T}_2(\mathbf{x}),\mathbf{f}_2(s),\mathbf{y}_2(\mathbf{x})\}$ be two equilibrium states, where $\mathbf{T}_{1(2)}$, $\mathbf{f}_{1(2)}$ are related to $\mathbf{y}_{1(2)}$ by the constitutive relations. According to eqn (105a), $E[\mathbf{y}_2]\geqslant E[\mathbf{y}_1]$ and $E[\mathbf{y}_1]\geqslant E[\mathbf{y}_2]$, since both $\mathbf{y}_1$ and $\mathbf{y}_2$ are kinematically admissible. Thus, $E[\mathbf{y}_1]=E[\mathbf{y}_2]$. Similar reasoning yields $E^*[\mathbf{T}_1,\mathbf{f}_1]=E^*[\mathbf{T}_2,\mathbf{f}_2]$. Now either equilibrium state satisfies (104b), so the results just derived yield $E^*[\mathbf{T}_2,\mathbf{f}_2]+E[\mathbf{y}_1]=0$, for example, and this requires that $\{\mathbf{T}_2,\mathbf{f}_2\}$ be related to $\mathbf{y}_1$ by the constitutive relations almost everywhere. Thus, $\mathbf{T}_2=\mathbf{T}_1$, $\mathbf{f}_2=\mathbf{f}_1$ and uniqueness is proved. The material regions in which the membrane and cable are tense are also uniquely determined. The same conclusions apply to the type-2 problem with obvious modifications.

The deformation need not be unique, however, as the constitutive equations (70) and (84) do not associate unique gradients $\mathbf{F}$ and $\mathbf{y}'$ with given values of $\mathbf{T}$ and $\mathbf{f}$, respectively. Partial uniqueness can be demonstrated if the relaxed energies are strictly convex on the first branches of eqns (74) and (85). The gradients are then uniquely determined, and integration determines the deformation uniquely up to a rigid translation (Cannarozzi, 1985; Pipkin, 1994; Atai and Steigmann, 1997). Weaker uniqueness results have been proved for deformations that generate strains in the 2nd or 3rd subdomains of the function eqn (74) (Pipkin, 1994). It is obvious that the deformation may be highly non-unique in slack regions where $\mathbf{T}$ or $\mathbf{f}$ vanish. Finally, we note that the results of this section are not strictly applicable to the standard linear solid because eqns (80), (81) do not define a meaningful relaxed energy for all strains.

5. NUMERICAL SOLUTION

5.1. *Dynamic relaxation*

It follows from the analysis of the previous section that energy minimizers and equilibria are essentially equivalent, provided that minimizing deformations are sufficiently regular and the relaxed strain energies are convex. Thus, we treat the equilibrium equations directly, rather than resort to a direct minimization procedure. Unfortunately, conditions guaranteeing the existence of minimizers do not, in general, yield the required regularity. Moreover, additional conditions ensuring only that minimizers are weak solutions of the equilibrium equations [satisfying eqn (27)] are not applicable to the present theory, even in its relaxed form (Ball, 1977). Thus, the computations described in Section 6 below should be regarded as numerical experiments.

The method we use, known as *dynamic relaxation*, is similar to the artificial viscosity schemes used by Silling (1987; 1988a, b; 1989), Swart and Holmes (1992), and Klouček and Luskin (1994) in applications involving non-convex theories of elasticity. These authors were mainly interested in simulating the dynamic evolution and asymptotic behavior of fine-scale features similar to those associated with minimizing sequences in the corresponding variational theories. Accordingly, they did not use relaxed strain energies in their computations. In applications to membrane theory, the method was used by Haseganu and Steigmann (1994a, 1996) as a means to overcome the ill-conditioning of conventional stiffness-based formulations derived from eqn (74).

In one version of the method the equilibrium equations for the membrane are replaced, for purposes of computation, by the artificial dynamical problem

$$\operatorname{div}\mathbf{T}(\mathbf{x},t) = \rho(\mathbf{x})\,\partial\mathbf{v}(\mathbf{x},t)/\partial t + c\rho(\mathbf{x})\mathbf{v}(\mathbf{x},t); \quad \mathbf{v}(\mathbf{x},t) = \partial\mathbf{y}(\mathbf{x},t)/\partial t \tag{110a}$$

subject to the *kinematically admissible* initial conditions

$$\mathbf{y}(\mathbf{x},0) = \mathbf{Y}(\mathbf{x}), \quad \mathbf{v}(\mathbf{x},0) = \mathbf{0}. \tag{110b}$$

Here $\rho(\mathbf{x})(>0)$ is the mass per unit area of the reference plane, c is a positive damping coefficient, t is the time, and $\mathbf{T}(\mathbf{x},t) = \hat{\mathbf{T}}(\operatorname{grad}\mathbf{y}(\mathbf{x},t))$.

Alternative formulations in which viscosity is introduced via constitutive equations are described in (Rybka, 1992; Swart and Holmes, 1992; Klouček and Luskin, 1994). We note that the latter equations are typically not frame-indifferent. However, as with eqn (110), this is irrelevant to the study of equilibria if solutions of the dynamical equations can be shown to decay asymptotically, and if the equilibria obtained are independent of the initial data and the particular viscosity formulation adopted. An outstanding difficulty with such methods is that equilibria typically *are* sensitive to the choice of formulation, at least in the non-convex studies cited. This sensitivity exists despite the path-independence of the *elastic* constitutive relations. The same kind of sensitivity, together with strong mesh dependence, was observed by Haseganu and Steigmann (1994a) in numerical experiments based on unrelaxed membrane strain energy functions, but no such sensitivity was found in computations based on relaxed (convex) strain energies. The insensitivity observed in the relaxed formulation is to be expected: the partial uniqueness result described in Section 4 implies that all equilibria have substantial features in common regardless of the methods used for their computation. It is thus arguable that, for the computation of equilibria, viscosity methods are more reliable in the present context than in the applications to non-convex problems considered elsewhere.

It remains to establish that equilibria are asymptotically stable in the class of dynamics described by eqn (110). The standard approach is to attempt to show that the potential energy is a positive–definite function of the deformation, with respect to a suitable norm, in some (hopefully large) neighborhood of an equilibrium configuration [e.g. Como and Grimaldi (1995)]. Asymptotic stability of the configuration is then assured if it can be demonstrated that the total mechanical energy is strictly decreasing on solutions of eqn (110). Application of this method to the present theory is hindered by the fact that the relaxed potential energy is only positive *semi*-definite (Section 4). Thus the question of asymptotic stability remains open. This deficiency is shared by the more sophisticated approaches described in (Rybka, 1992; Swart and Holmes, 1992).

However, it is possible to demonstrate that the total energy is non-increasing on smooth solutions of eqn (110). [Rybka (1992) has established the smoothening effect of viscosity in a similar context.] To see this we scalar-multiply the latter equation by $\mathbf{v}$, integrate over the reference plane and use eqn (70) to derive

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[K+\int_{\Omega} W(\mathbf{F})\,\mathrm{d}a\right]+c\int_{\Omega}\rho|\mathbf{v}|^2\,\mathrm{d}a=\int_{\partial\Omega}\mathbf{Tv}\cdot\mathbf{v}\,\mathrm{d}s, \tag{111}$$

where

$$K=\frac{1}{2}\int_{\Omega}\rho|\mathbf{v}|^2\,\mathrm{d}a \tag{112}$$

is the kinetic energy and W is the relaxed membrane strain energy. Coupling with the attached cable(s) may be incorporated by using eqn (23) to write

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} W(\mathbf{F})\,\mathrm{d}a=\frac{\mathrm{d}}{\mathrm{d}t}(E-U). \tag{113}$$

For the type-1 problem, a formula analogous to eqn (28) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}U=\int_{P}\mathbf{f}\cdot\mathbf{v}'\,\mathrm{d}s=\sum[\mathbf{f}\cdot\mathbf{v}]_{\partial P_{\Gamma}}-\int_{P}\mathbf{f}'\cdot\mathbf{v}\,\mathrm{d}s. \tag{114}$$

This also applies to the type-2 problem if $\mathbf{f}$ is replaced by $f\mathbf{t}$, with f independent of s, and if the endpoint terms are dropped. On combining eqns (111)–(114) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(K+E)+c\int_{\Omega}\rho|\mathbf{v}|^2\,\mathrm{d}a=\sum[\mathbf{f}\cdot\mathbf{v}]_{\partial P_{\Gamma}}+\int_{P}(\mathbf{Tv}-\mathbf{f}')\cdot\mathbf{v}\,\mathrm{d}s+\int_{\partial\Omega\backslash P}\mathbf{Tv}\cdot\mathbf{v}\,\mathrm{d}s. \tag{115}$$

In the numerical method described below we impose conditions (30), (32), (35) and (31) or (41), as appropriate, at all values of the time. This is tantamount to neglecting the mass (and viscosity) of the cable. Then the right-hand side of eqn (115) vanishes, yielding

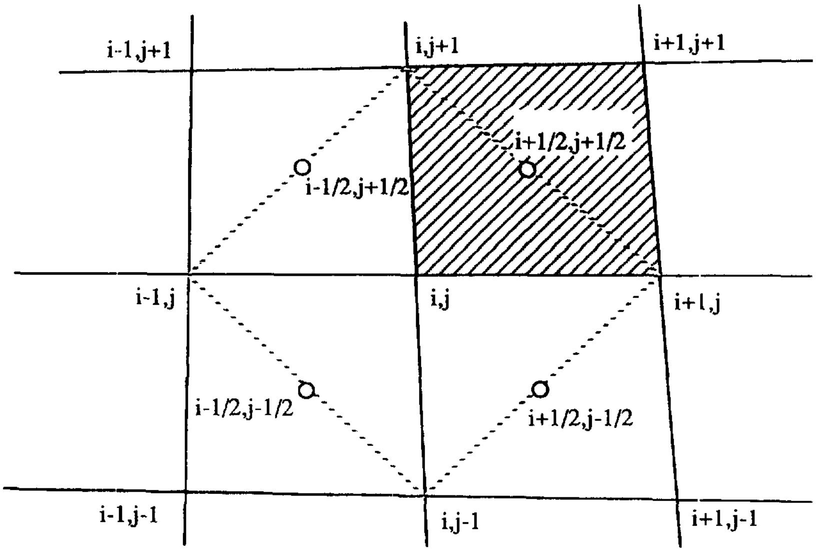

Fig. 1. A unit cell of the finite difference mesh.

$$\frac{d}{dt}(K+E) \leqslant 0, \tag{116}$$

provided that $c > 0$. This result and the minimizing property of equilibria motivate the numerical experiments described in Section 6.

5.2. *Finite-difference scheme*

We use a variant of the Green's theorem spatial discretization method described by Silling (1988b). The adaptation of this method to membrane theory is straightforward and is discussed fully in (Haseganu and Steigmann, 1994a). Thus, we simply summarize the main formulae, with emphasis on the modifications needed to accommodate suturing and cable–membrane interactions.

The reference plane Ω is covered with a grid consisting of cells of the type shown in Fig. 1. Nodes are identified by integer superscripts (i,j); $x_\alpha^{i,j}$ are the reference values of the Cartesian coordinates of node (i,j). The regions formed by the four nearest neighbors of a node are called *zones*. Zone-centered points, indicated by open circles in the figure, are labelled with half-integer superscripts.

The divergence of the Piola stress at node (i,j) is obtained by applying Green's theorem to the quadrilateral region enclosed by the dashed contour. The area integral is estimated as the nodal value of the integrand times the enclosed area, while the contour integral is approximated by setting its integrand equal to the zone-centered values on each of the four edges that make up the boundary. This yields a difference formula for the net force $\mathbf{P}^{i,j} = P_k^{i,j}\mathbf{e}_k$ acting on the node:

$$P_k^{i,j} = 2A^{i,j}(T_{k\alpha,\alpha})^{i,j} = e_{\alpha\beta}[T_{k\alpha}^{i+1/2,j+1/2}(x_\beta^{i,j-1} - x_\beta^{i+1,j}) + T_{k\alpha}^{i-1/2,j+1/2}(x_\beta^{i-1,j} - x_\beta^{i,j+1}) + T_{k\alpha}^{i-1/2,j-1/2}(x_\beta^{i,j-1} - x_\beta^{i-1,j}) + T_{k\alpha}^{i+1/2,j-1/2}(x_\beta^{i+1,j} - x_\beta^{i,j-1})], \tag{117}$$

where $e_{\alpha\beta}$ is the unit alternator ($e_{12} = -e_{21} = 1$, $e_{11} = e_{22} = 0$) and $A^{i,j}$ is the area of half the quadrilateral

$$A^{i,j} = \tfrac{1}{4}[(x_2^{i-1,j} - x_2^{i+1,j})(x_1^{i,j+1} - x_1^{i,j-1}) - (x_1^{i-1,j} - x_1^{i+1,j})(x_2^{i,j+1} - x_2^{i,j-1})]. \tag{118}$$

Equation (117) is derived by using the relation $\boldsymbol{\nu}\,ds = d\mathbf{x} \times \mathbf{e}_3$ on the boundary.

The stresses on the right-hand side of eqn (117) are evaluated at zone-centered points. These depend via the constitutive equations on the zone-centered values of the deformation

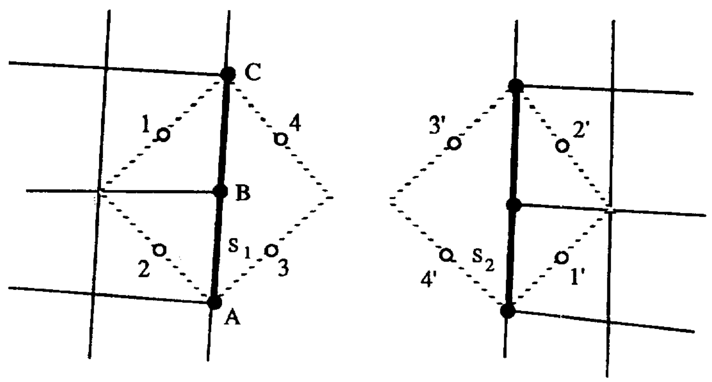

Fig. 2. Suturing of two membrane edges.

gradient. We compute these values by applying Green's theorem to the shaded region shown in Fig. 1. As before, the associated area integral is estimated as the integrand at the zone-centered point times the shaded area. However, the four edge contributions to the boundary integral are now approximated by replacing the integrand in each with the average of the nodal values at the endpoints. The resulting difference formula is (Silling, 1988b):

$$F_{k\alpha}^{i+1/2,j+1/2} = (2A^{i+1/2,j+1/2})^{-1}\, e_{\alpha\beta}[(x_\beta^{i,j+1} - x_\beta^{i+1,j})(y_k^{i+1,j+1} - y_k^{i,j}) - (x_\beta^{i+1,j+1} - x_\beta^{i,j})(y_k^{i,j+1} - y_k^{i+1,j})], \quad (119)$$

where

$$A^{i+1/2,j+1/2} = \tfrac{1}{2}[(x_2^{i,j+1} - x_2^{i+1,j})(x_1^{i+1,j+1} - x_1^{i,j}) - (x_1^{i,j+1} - x_1^{i+1,j})(x_2^{i+1,j+1} - x_2^{i,j})]. \quad (120)$$

Stresses at zone-centered points may now be obtained from the components of the deformation gradient according to the procedure described in (Haseganu and Steigmann, 1994a). The truncation errors associated with eqns (117) and (119) are discussed in (Silling, 1988b) and (Herrmann and Bertholf, 1983).

Traction-free boundaries are simulated by distributing nodes along the boundary and setting the stresses at the zone-centered points exterior to the domain equal to zero. This approximates the traction data if a sufficiently fine mesh is used. A similar procedure may be used to simulate suturing. Thus, consider two edges s_1 and s_2 that are to be sutured together (Fig. 2). These edges are chosen so as to span three nodes each. We then approximate the terms in eqn (36) by

$$\int_{s_1} \mathbf{T}\mathbf{v}\,ds \simeq \int_3 \mathbf{T}\mathbf{v}\,ds + \int_4 \mathbf{T}\mathbf{v}\,ds, \quad \int_{s_2} \mathbf{T}\mathbf{v}\,ds \simeq \int_{3'} \mathbf{T}\mathbf{v}\,ds + \int_{4'} \mathbf{T}\mathbf{v}\,ds \quad (121)$$

wherein subscripts 3(3′) and 4(4′) refer to the two dashed segments exterior to the left-(right-)hand part of the membrane shown in the figure, and the integration paths are traversed in accordance with the requirements of Green's theorem. Equation (36) is approximated by requiring that the sum of the integrals over segments 3, 3′, 4, and 4′ be zero. This condition may be incorporated into the procedure for conventional nodes by applying Green's theorem to each of the dashed contours. Adding the resulting expressions and invoking the approximation to eqn (36), we derive

$$\int_{\Omega_L} \operatorname{div}\mathbf{T}\,da + \int_{\Omega_R} \operatorname{div}\mathbf{T}\,da \simeq \sum_{i=1}^{2} \int_i \mathbf{T}\mathbf{v}\,ds + \sum_{i'=1}^{2} \int_{i'} \mathbf{T}\mathbf{v}\,ds, \tag{122}$$

where Ω_L and Ω_R are the regions enclosed by the left and right contours, respectively. The zone-centered points exterior to the two boundaries are then collapsed onto the boundaries, so that Ω_L, Ω_R reduce to the regions enclosed by the boundaries and the dashed interior segments that remain.

The area-weighted average of div **T** furnished by the two contours is equal to the left-hand side of eqn (122) divided by $A_L + A_R$, the total area of the region $\Omega_L \cup \Omega_R$. This average value is assigned to each member of the pair of opposing nodes along the two edges to be sutured. The resulting difference formula is identical to eqn (117) (apart from labelling) with the stresses on the right-hand side of the second equality evaluated at the interior zone-centered points along segments 1, 2, 1′ and 2′, and with $2A$ on the left-hand side of that equality replaced by $A_L + A_R$. With this interpretation of div **T**, the first equality in eqn (117) furnishes the net force on the common deformed node. All that remains is to assign the same *deformed* position **y** to the opposing nodes.

Cable–membrane interaction may be approximated in a similar manner. Thus suppose that a cable is attached to the boundary of the left-hand region in Fig. 2. The net force on the boundary node B of the membrane is again given by

$$\mathbf{P} = \int_{\Omega_L} \operatorname{div}\mathbf{T}\,da = \sum_{i=1}^{4} \int_i \mathbf{T}\mathbf{v}\,ds. \tag{123}$$

The sum of the integrals over segments 3 and 4 approximates the integral of the traction applied to the membrane from point A to point C along the edge. According to the coupling condition, eqn (31), this is equal to $\mathbf{f}_C - \mathbf{f}_A$, where $\mathbf{f}_{A(C)}$ is the force in the part AB (resp. BC) of the cable, evaluated at node A (resp. C). Thus

$$\mathbf{P} \simeq \sum_{i=1}^{2} \int_i \mathbf{T}\mathbf{v}\,ds + \mathbf{f}_C - \mathbf{f}_A. \tag{124}$$

The traction integrals over the segments 1, 2 in the interior of the membrane are estimated by replacing the stresses with their values at the zone-centered points, as in eqn (117). To obtain the cable forces we consider the cable to be attached to the membrane only at the nodes. The actual cable is thus replaced by a connected sequence of cables. This approximates the continuous attachment condition if a sufficiently fine mesh is used. Locally, the cables between the nodes now respond as though the distributed loads acting on them vanished. Equations (7) and (8) then imply that the deformed cables are locally straight and uniformly stressed. We are neglecting the masses of the cables. For the type-1 problem, this leads to the approximations

$$\mathbf{f}_C = f(\lambda_{BC})\mathbf{t}_{BC}, \quad \mathbf{f}_A = f(\lambda_{AB})\mathbf{t}_{AB} \tag{125a}$$

where $f(\cdot)$ is the force–stretch relation of the actual cable, presumed to be elastically uniform,

$$\mathbf{t}_{BC} = (\mathbf{y}_C - \mathbf{y}_B)/|\mathbf{y}_C - \mathbf{y}_B| \quad \text{and} \quad \mathbf{t}_{AB} = (\mathbf{y}_B - \mathbf{y}_A)/|\mathbf{y}_B - \mathbf{y}_A| \tag{125b}$$

are the directions of the segments between the deformed nodes, $\mathbf{y}_{A,B,C}$ are the nodal positions, and

$$\lambda_{BC} = |\mathbf{y}_C - \mathbf{y}_B|/|\mathbf{x}_C - \mathbf{x}_B|, \quad \lambda_{AB} = |\mathbf{y}_B - \mathbf{y}_A|/|\mathbf{x}_B - \mathbf{x}_A| \tag{125c}$$

are the stretches. The type-2 problem is handled in the same way except that the stretches are now taken to be equal to the total arclength of the deformed cable divided by the total reference arclength:

$$\lambda_{AB} = \lambda_{BC} = l/L, \tag{125d}$$

where l and L are estimated as the sums of the straight-line distances between successive boundary nodes in the deformed and reference configurations, respectively. A combination of the foregoing procedures may be used to simulate the practically important problem of a cable attached to a suture line, but we do not consider such examples here.

We remark that preliminary calculations for several of the examples discussed in Section 6 below exhibited spurious zero-energy modes, or *hourglassing*, which typically results in serious degradation of the solution. Special measures to suppress hourglassing have been described by Flanagan and Belytschko (1981) and Silling (1988b). Rather than adopt such measures here, we have instead used hourglassing to assess the suitability of a particular mesh topology for the problem under study. Wherever the effect was encountered, the mesh was redesigned and the calculations repeated. The final mesh designs shown in this work did not exhibit hourglassing.

Equilibrium of node (i,j) requires that $\mathbf{P}^{i,j} = \mathbf{0}$ unless the node is constrained. We solve these equations by the method of dynamic relaxation. The discrete form of the equations is obtained by integrating eqn (110a) over the quadrilateral region formed by the four nearest neighbors of the node and setting the deformation in the interior of the region equal to the nodal value. Thus

$$M^{i,j}\ddot{\mathbf{y}}^{i,j,n} + cM^{i,j}\dot{\mathbf{y}}^{i,j,n} = \mathbf{P}^{i,j,n}, \tag{126a}$$

where the superscript n identifies the time value t_n, and

$$M^{i,j} = 2A^{i,j}\rho(\mathbf{x}^{i,j}) \tag{126b}$$

is the nodal mass. The time derivatives are replaced by the central differences

$$\dot{\mathbf{y}}^n = \frac{1}{2}(\dot{\mathbf{y}}^{n+1/2} + \dot{\mathbf{y}}^{n-1/2}), \quad \ddot{\mathbf{y}}^n = \frac{1}{h}(\dot{\mathbf{y}}^{n+1/2} - \dot{\mathbf{y}}^{n-1/2}), \quad \dot{\mathbf{y}}^{n-1/2} = \frac{1}{h}(\mathbf{y}^n - \mathbf{y}^{n-1}), \tag{127}$$

where h is the time increment and the node label (i,j) has been suppressed. These are accurate to order $O(h)$. On substituting them into eqn (126a) we obtain the explicit decoupled system (Underwood, 1983; Haseganu and Steigmann, 1994a):

$$\begin{aligned}(h^{-1} + c/2)M^{i,j}\dot{\mathbf{y}}^{i,j,n+1/2} &= (h^{-1} - c/2)M^{i,j}\dot{\mathbf{y}}^{i,j,n-1/2} + \mathbf{P}^{i,j,n},\\ \mathbf{y}^{i,j,n+1} &= \mathbf{y}^{i,j,n} + h\dot{\mathbf{y}}^{i,j,n+1/2},\end{aligned} \tag{128}$$

which is used to advance the solution in time node by node.

We remark that a formulation in which viscosity is introduced through the constitutive relations rather than through modification of the equations of motion [cf. eqn (110a)] would, in general, result in a coupled system for the updated values of the nodal position vectors.

Silling (1987) demonstrated the conditional stability of a scheme similar to the one adopted here by performing a linear stability analysis for a comparable one-dimensional model problem. In the present work we have used numerical experiments to find combinations of nodal mass and damping that furnish stable calculations decaying rapidly to equilibrium. We have made no effort to optimize these combinations, however. An adaptive method for choosing optimal or near-optimal combinations may be found in (Papadrakakis, 1981).

The starting procedure for eqn (128) is derived from eqn (110b). Thus we prescribe $\mathbf{y}^{i,j,0} = \mathbf{Y}(\mathbf{x}^{i,j})$ and set $\dot{\mathbf{y}}^{i,j,0} = \mathbf{0}$. It follows from eqns (127) and (128) that

$$(2/h)M^{i,j}\dot{\mathbf{y}}^{i,j,1/2} = \mathbf{P}^{i,j,0}, \tag{129}$$

wherein the right-hand side is determined by $\mathbf{Y}(\mathbf{x})$. The system of equations is non-dimensionalized by the product of the shear modulus $\mu(>0)$ [cf. eqns (77)–(79)] and a characteristic length, and the solution is advanced to the first t_n that satisfies $\max_{(i,j)}|\mathbf{P}^{i,j,n}| < 10^{-5}$.

6. EXAMPLES

We discuss the results obtained by applying the dynamic relaxation procedure to a number of illustrative problems. We first describe examples for which analytical solutions are readily available [e.g. Carroll (1988)] and then present comparisons with numerical results. This is followed by discussions of a number of boundary value problems for which no analytical results are known. In all examples the use of various alternative initial conditions in the artificial dynamical problem has furnished final equilibrium configurations that are essentially indistinguishable. The positions of the deformed nodes are also converged with respect to mesh refinement.

6.1. *Plane deformations of annular membranes*

We consider the class of axisymmetric radial plane deformations

$$\mathbf{x} = r\mathbf{e}_r(\theta) \rightarrow \mathbf{y} = \rho(r)\mathbf{e}_r(\theta), \tag{130}$$

where (r, θ) are polar coordinates in the reference plane and $\mathbf{e}_r(\theta)$ is a unit vector directed along the ray $\theta = \text{const.}$

In a general plane deformation, not necessarily of the form (130), the matrix of the deformation gradient is square and its determinant is well defined. The two-dimensional polar decomposition theorem may then be applied to obtain

$$\mathbf{F} = \lambda_1\mathbf{v}_1 \otimes \mathbf{u}_1 + \lambda_2\mathbf{v}_2 \otimes \mathbf{u}_2, \tag{131}$$

where λ_1, λ_2 are the principal stretches, $\mathbf{u}_1$, $\mathbf{u}_2$ are the orthonormal principal vectors of $\mathbf{F}^T\mathbf{F}$ [cf. eqn (13)], and $\mathbf{v}_1$, $\mathbf{v}_2$ are the orthonormal principal vectors of $\mathbf{F}\mathbf{F}^T$. The determinant of $\mathbf{F}$ is $J = \lambda_1\lambda_2$ [cf. eqn (79)]. From eqns (17) and (21) it then follows that the Piola stress admits the representation

$$\mathbf{T} = w_1\mathbf{v}_1 \otimes \mathbf{u}_1 + w_2\mathbf{v}_2 \otimes \mathbf{u}_2. \tag{132}$$

This remains valid for three-dimensional deformations also, with $\mathbf{v}_1$, $\mathbf{v}_2$ spanning the tangent plane of the deformed surface at the point where $\mathbf{T}$ is evaluated.

We note that the relations (79b, c) defining I and J in terms of the stretches are invertible. Thus the strain energy may be represented by a function $W(I, J)$ (say), and

$$w_1 = W_I + \lambda_2 W_J, \quad w_2 = W_I + \lambda_1 W_J. \tag{133}$$

Substitution into eqn (132) furnishes the result

$$\mathbf{T} = I^{-1}W_I(\mathbf{F}+\mathbf{F}^*) + W_J\mathbf{F}^*, \tag{134}$$

where $\mathbf{F}^*$ is the *adjugate* of $\mathbf{F}$:

$$\mathbf{F}^* = J\mathbf{F}^{-T} = \lambda_2 \mathbf{v}_1 \otimes \mathbf{u}_1 + \lambda_1 \mathbf{v}_2 \otimes \mathbf{u}_2. \tag{135}$$

The identity $\operatorname{div}\mathbf{F}^* = \mathbf{0}$ makes eqn (134) attractive for the analysis of the equilibrium equation.

For deformations of the class eqn (130), the deformation gradient is

$$\mathbf{F} = \rho'(r)\mathbf{e}_r \otimes \mathbf{e}_r + (\rho/r)\mathbf{e}_\theta \otimes \mathbf{e}_\theta, \tag{136}$$

where $\mathbf{e}_\theta(\theta) = \mathbf{e}_3 \times \mathbf{e}_r$. The requirement that $J = (\rho/r)\rho'$ be positive yields $\rho' > 0$ and the stretches are $\lambda_1 = \rho'$, $\lambda_2 = \rho/r$. The adjugate is

$$\mathbf{F}^* = (\rho/r)\mathbf{e}_r \otimes \mathbf{e}_r + \rho'(r)\mathbf{e}_\theta \otimes \mathbf{e}_\theta, \tag{137}$$

and we note that

$$\mathbf{F} + \mathbf{F}^* = I\mathbf{1}; \quad I = \rho' + \rho/r, \tag{138}$$

where $\mathbf{1}$ is the two-dimensional unit tensor. Then eqn (134) reduces to

$$\mathbf{T} = W_I \mathbf{1} + W_J \mathbf{F}^* \tag{139}$$

for axisymmetric radial deformations.

Suppose now that an elastically uniform cable is attached to the membrane along its inner edge $r = r_i$. Its stretch is $\lambda = \lambda_2(r_i)$ and its direction is $\mathbf{t}(s) = -\mathbf{e}_\theta(\theta)$, where $\theta = -s/r_i$. (Recall that s is measured in accordance with Green's theorem; it runs clockwise along the inner boundary of the annulus.) Thus the force transmitted by the cable is

$$\mathbf{f}(s) = -f(\lambda_2(r_i))\mathbf{e}_\theta. \tag{140}$$

This is valid for both types of cable-membrane interaction. On substituting this into eqn (31) [or eqn (41)] with $\mathbf{v} = -\mathbf{e}_r$, and invoking eqns (137)–(139), we obtain the coupling condition

$$f/r_i = W_I + (\rho/r) W_J, \tag{141}$$

with the right-hand side evaluated at $r = r_i$. At the outer boundary $r = r_0$ we prescribe

$$\rho(r_0) = \rho_0 (> r_0), \tag{142}$$

and solve the associated boundary value problem for two membrane strain energy functions.

6.1.1. *Varga material.* We assume that (142) generates biaxial tension everywhere in the annulus and thus that the stretches belong to the first branch of the relaxed Varga strain energy defined by eqns (74) and (78). We further assume that the cable stretch exceeds unity. The assumptions are verified after the solution is obtained. Thus, $W_I = 2\mu$ (= const.), $W_J = -2\mu/J^2$ and the equilibrium equation (19), with $\mathbf{T}$ given by eqn (139), reduces to $\operatorname{grad} J = \mathbf{0}$ (Varga, 1966). The associated deformation is

$$\rho(r) = [\rho_0^2 - J(r_0^2 - r^2)]^{1/2}, \quad J = \text{const}, \tag{143}$$

and the coupling condition (141) yields:

$$f(\lambda_2)/r_i = 2\mu(1 - \lambda_2/J^2), \tag{144}$$

where

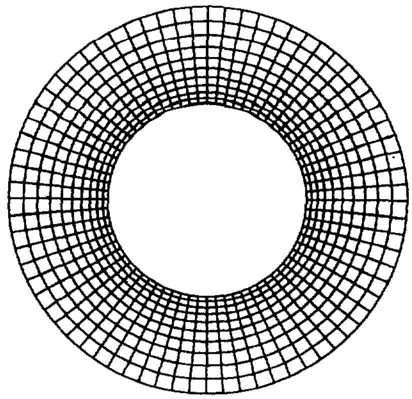

Fig. 3. Meshed reference configuration of an annular membrane ($r_i/r_0 = 0.5$).

$$\lambda_2 = \lambda_2(r_i) = \left\{\left(\frac{\rho_0}{r_i}\right)^2 - J\left[\left(\frac{r_0}{r_i}\right)^2 - 1\right]\right\}^{1/2}. \tag{145}$$

We solve eqn (144) for $\lambda_2(r_i)$ and substitute the result into eqn (145) to derive an equation for J. For the cable force–stretch relation given by eqn (86), the solution of eqn (144) is:

$$\lambda_2(r_i) = \left(1 + \frac{E}{2\mu r_i}\right) \Big/ \left(J^{-2} + \frac{E}{2\mu r_i}\right). \tag{146}$$

Equating this with eqn (145) yields an equation for J which may be solved by Newton's method, for example. With J known, the solution is then completely determined.

A comparison of the analytical and numerical solutions is made for the case

$$r_i/r_0 = 0.5, \quad \rho_0/r_0 = 1.2, \tag{147}$$

and the radius r is non-dimensionalized by r_0. Figure 3 shows the mesh used for the computations. This consists of 11 radial nodes distributed along each of 72 equally spaced rays. No symmetry conditions are imposed.

Figure 4(a) shows the distributions of the radial stretch λ_1 and the hoop stretch λ_2 for the case $E = 0$ (no cable). These are calculated at zone-centered points along rays and represented by triangles and circles, respectively. The numerical solution possesses the expected axisymmetry. The solid curves are obtained from the analytical solution. The stretch distributions for the case $E/2\mu r_i = 2.0$ are given in Fig. 4(b), in which the vertical scale has been magnified. The results indicate that the principal effect of the cable is to strongly suppress the extreme values of the stretches in the membrane at the inner boundary. In both examples the stretches fall within the range in which the Varga material furnishes quantitative agreement with biaxial data on rubber (Varga, 1966). The *a priori* assumptions regarding the stretches are also readily verified.

6.1.2. *Harmonic materials.* We solve the same boundary value problem for the class of harmonic materials defined by eqn (80). Thus, $W_I = 2\mu(I-1)+\bar{\lambda}(I-2)$, $W_J = -2\mu$ and the equilibrium equation becomes $(\bar{\lambda}+2\mu)\,\mathrm{grad}\, I = \mathbf{0}$. If $\bar{\lambda}+2\mu \neq 0$ then the associated deformation is

$$\rho(r) = \frac{I}{2}(r - r_0^2/r) + \rho_0 r_0/r; \quad I = \text{const}, \tag{148}$$

and the coupling condition (141) reduces to

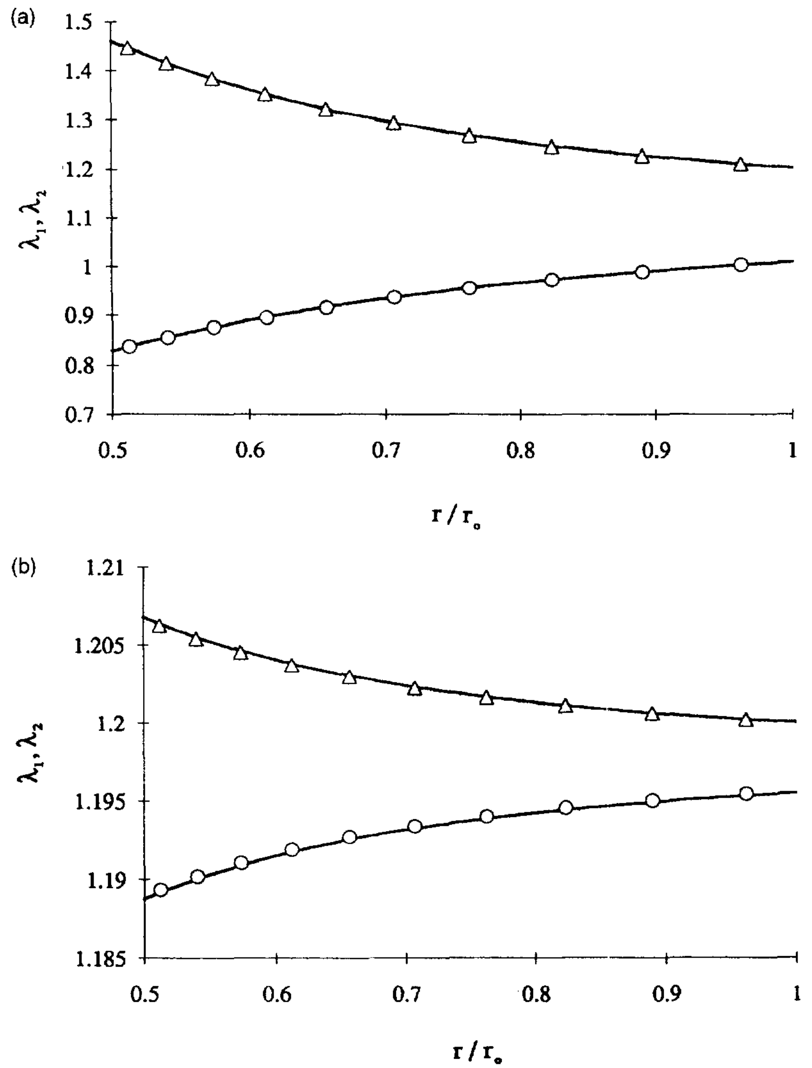

Fig. 4. Principal stretch distributions λ_1 (○) and λ_2 (△) in an annular Varga membrane with $\rho_0/r_0 = 1.2$. The solid curves represent the analytical distributions: (a) no cable; (b) cable with modulus $E/2\mu r_i = 2.0$.

$$2\mu(I-1)+\bar{\lambda}(I-2) = (2\mu+E/r_i)\lambda_2 - E/r_i, \tag{149}$$

where

$$\lambda_2 = \lambda_2(r_i) = \frac{I}{2}(1-r_0^2/r_i^2)+\rho_0 r_0/r_i^2. \tag{150}$$

The combination of these formulae yields a linear equation for the value of I.

To obtain quantitative results we use the data (147) and set $\bar{\lambda}/\mu = 2.0$. Figure 5(a) shows the stretch distributions obtained for the case $E = 0$ (no cable) using the mesh of Fig. 3. The effect of an attached cable with $E/2\mu r_i = 2.0$ is shown in Fig. 5(b). The solutions are seen to be in close agreement with the predictions of the Varga model.

6.2. *Neutral holes*

In the problem of a solid disc subject to eqn (142), the uniform equibiaxial stretch with gradient

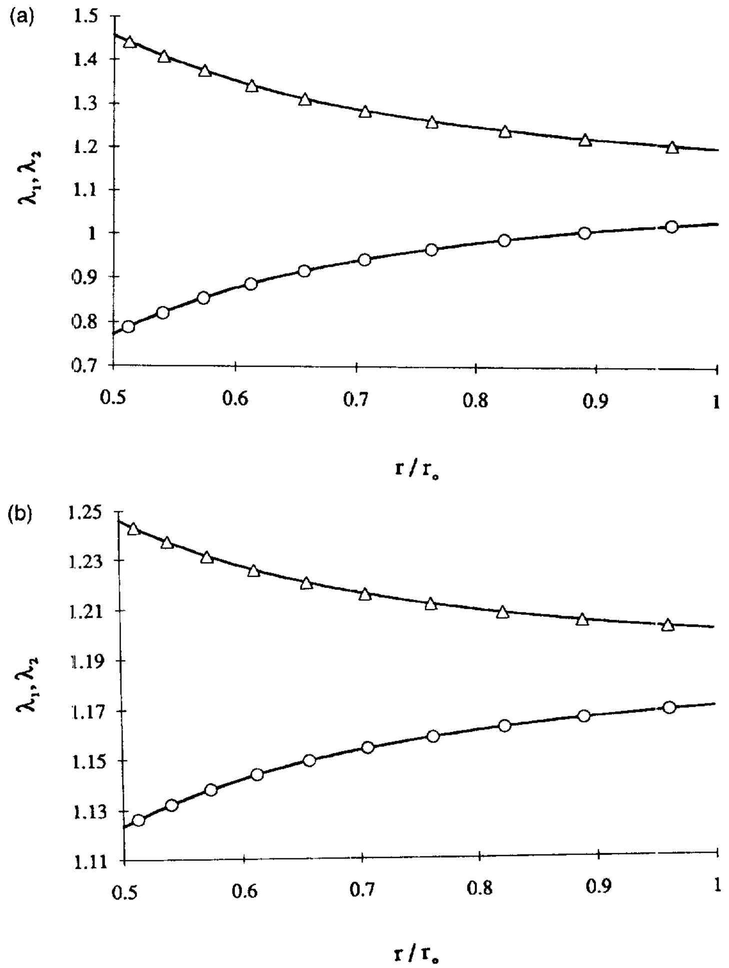

Fig. 5. Principal stretch distributions λ_1 (○) and λ_2 (△) in the standard linear solid ($\bar{\lambda}/\mu = 2.0$) with $\rho_0/r_0 = 1.2$. The analytical distributions are represented by the solid curves: (a) no cable; (b) cable with modulus $E/2\mu r_i = 2.0$.

$$\mathbf{F}(\mathbf{x}) = \lambda \mathbf{1}\,; \quad \lambda = \rho_0/r_0(>1) \tag{151}$$

minimizes the relaxed potential energy absolutely, and is thus a stable equilibrium state according to the energy criterion. This is due to the fact that the quasi-convexity condition (49), with $D = \Omega$, reduces, in this instance, to the minimum energy inequality (105a). In the setting of the classical linear theory of elasticity for infinitesimal displacements, Mansfield (1953) demonstrated the remarkable result that a uniform equibiaxial strain furnishes an equilibrium state for the annulus also, provided that a cable with certain specific elastic properties is attached to the inner boundary. The effect of the cable is then to completely neutralize the strain gradient in the annulus that would otherwise exist. Mansfield extended the result to a number of hole shapes and obtained the required reinforcement properties in each case.

Here we show that, in the presence of finite deformations, the annulus problem for isotropic materials admits solutions with gradient of the form (151) only if a certain restrictive condition is met by the membrane and cable response functions. When specialized to infinitesimal strains, the restriction amounts to an algebraic relation among the elastic moduli, in agreement with Mansfield's result.

The adjugate of eqn (151) is $\mathbf{F}^* = \mathbf{F}$, and eqn (139) reduces to

$$\mathbf{T} = T(\lambda)\mathbf{1}, \quad \text{where } T(\lambda) = W_I + \lambda W_J \tag{152}$$

and the derivatives of $W(I, J)$ are evaluated at $I = 2\lambda$ and $J = \lambda^2$. The equilibrium equation is satisfied identically and the coupling condition at the inner boundary $r = r_i$ is

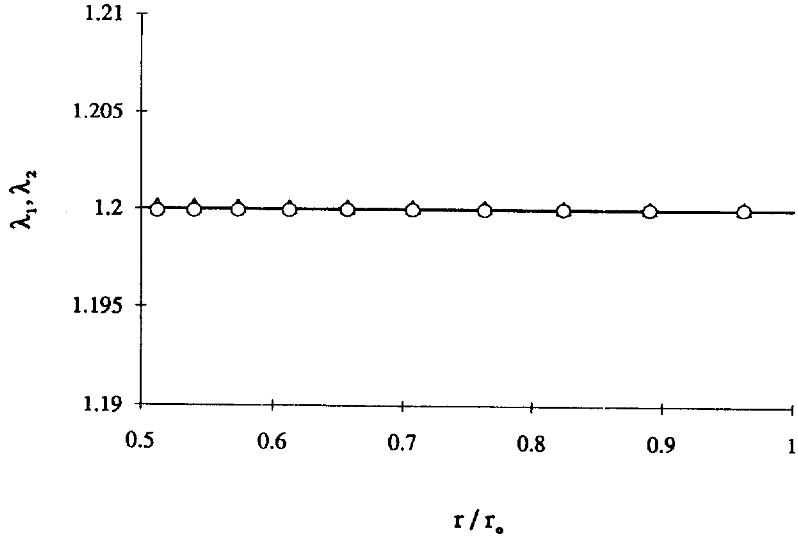

Fig. 6. Neutral hole in the standard linear solid ($\bar{\lambda}/\mu = 2.0$) with $\rho_0/r_0 = 1.2$ and cable modulus $E/2\mu r_i = 3.0$. The labelling used is as in Figs 4 and 5.

$$f(\lambda)/r_i = T(\lambda). \tag{153}$$

This is *not* satisfied by typical response functions $f(\cdot)$ and $T(\cdot)$. However, it may be satisfied approximately at small strains if the cable and membrane are unstressed in their reference configurations. To see this we write

$$f(\lambda) = E(\lambda-1)+o(\varepsilon), \quad T(\lambda) = 2(\bar{\lambda}+\mu)(\lambda-1)+o(\varepsilon), \tag{154}$$

where $\varepsilon = \lambda - 1$ and E, $\bar{\lambda}$, μ are the values of the various moduli in the reference configuration. Then eqn (153) is satisfied to leading order if

$$E/r_i = 2(\bar{\lambda}+\mu). \tag{155}$$

For example, the small-strain approximation to the Varga strain energy yields $\bar{\lambda}/\mu = 2.0$ and eqn (155) furnishes the dimensionless cable modulus

$$\frac{E}{2\mu r_i} = 3.0. \tag{156}$$

Evidently the same moduli solve eqn (153) without restriction to small strains if eqn (86) is used as the cable response function and if the standard linear solid is used to model the membrane. We note that neutral holes are possible in materials of this class only if the local convexity conditions (82a, b) are satisfied. Figure 6 shows the computed and exact stretch distributions for the indicated values of the moduli and for the data given in eqn (147). These may be compared to those of Fig. 5(a, b). The numerical method furnishes a deformation that is seen to be very nearly homogeneous and equibiaxial.

6.3. *Plane deformation of a square sheet*

A striking example of cable–membrane interaction is furnished by the plane deformation of a square sheet produced by the displacement of its corners. The edges of the square are traction-free. We anticipate corner singularities in the stretches and strong stretch gradients in the vicinities of the corners. Behavior of this kind is not well described by any of the strain energy functions discussed in Section 3.3. Accordingly, we base our calculations on Ogden's three-term strain energy function, which is known to furnish quantitative agreement with data on rubber over a wide range of strain (Ogden, 1984).

Moreover, the large-strain behavior of the Ogden material is such that existence of solutions with finite energy can be expected even in the presence of strong singularities. Examples of this kind, for which alternative strain energies yield non-existence, are discussed in Li and Steigmann (1995).

In the present context the strain energy function is (Ogden, 1984):

$$w(\lambda_1,\lambda_2) = \mu \sum_{r=1}^{3} \beta_r[\lambda_1^{\alpha_r}+\lambda_2^{\alpha_r}+(\lambda_1\lambda_2)^{-\alpha_r}-3]/\alpha_r, \tag{157a}$$

where

$$\alpha_1 = 1.3,\quad \alpha_2 = 5.0,\quad \alpha_3 = -2.0;\quad \beta_1 = 1.491,\quad \beta_2 = 0.003,\quad \beta_3 = -0.0237. \tag{157b}$$

The procedure described in Section 3.3 furnishes the relaxed form (74) with (Li and Steigmann, 1995):

$$\hat{w}(x) = \mu \sum_{r=1}^{3} \beta_r(x^{\alpha_r}+2x^{-\alpha_r/2}-3)/\alpha_r,\quad v(x) = x^{-1/2}. \tag{158}$$

We did not list these formulae among the examples of Section 3.3 because their complexity precludes an analytical determination of the degree to which they comply with the convexity conditions used in the definition of eqn (74). A further complication is that the indicated function $v(x)$ is not the unique function meeting the conditions stipulated in Section 3 in this instance. It is obtained by invoking the bulk incompressibility of the actual material and assuming the (three-dimensional) stretches transverse to the axis of uniaxial tension to be equal prior to wrinkling. We conjecture that it furnishes the optimal version of eqn (74) for the problems considered.

Figure 7(a) shows the reference configuration overlaid with the mesh used for the computations. This consists of 861 nodes arranged in such a way as to resolve the strong gradients expected near the corners. The deformed mesh is shown in Fig. 7(b). The edges of the square are traction-free and the corner nodes are displaced diametrically so that the straight-line distances between adjacent corners are all equal to $1.5L$, where L is the length of a side in the reference configuration. The latter length scale is used to non-dimensionalize the equations. No symmetry conditions are imposed. The numerical method cannot resolve the expected stretch singularities at the corners because the stretches are evaluated at zone-centered points situated a small distance from the nodes. The largest computed principal stretch is 8.10 at the zone-centered points nearest the four corners. The minimum stretch is 1.04 at the zone-centered points closest to the center of the square. The membrane is found to be under local biaxial tension at all zone-centered points.

The deformation generated by cable–membrane interaction is shown in Fig. 7(c, d). In both cases the dimensionless cable modulus is $E/\mu L = 10.0$. The first figure corresponds to a type-1 problem in which the cables are fixed to the four edges of the membrane, and the second to a type-2 problem in which the cables slide freely. In both cases the effect of the cable is to strongly suppress strain gradients in the interior. The largest computed stretch in the type-1 problem is 1.67 near the corners and the minimum is 1.44 near the center. In the type-2 problem the extreme stretches occur at the same locations but their values are 2.47 and 1.42. The larger maximum in the second problem is to be expected because of the weaker interaction between the cable and membrane. The nodal deformations are nearly identical in the two problems despite localized discrepancies in strain. Both deformations are nearly homogeneous outside the immediate neighborhoods of the corners. Again the stress is local biaxial tension at all zone-centered points.

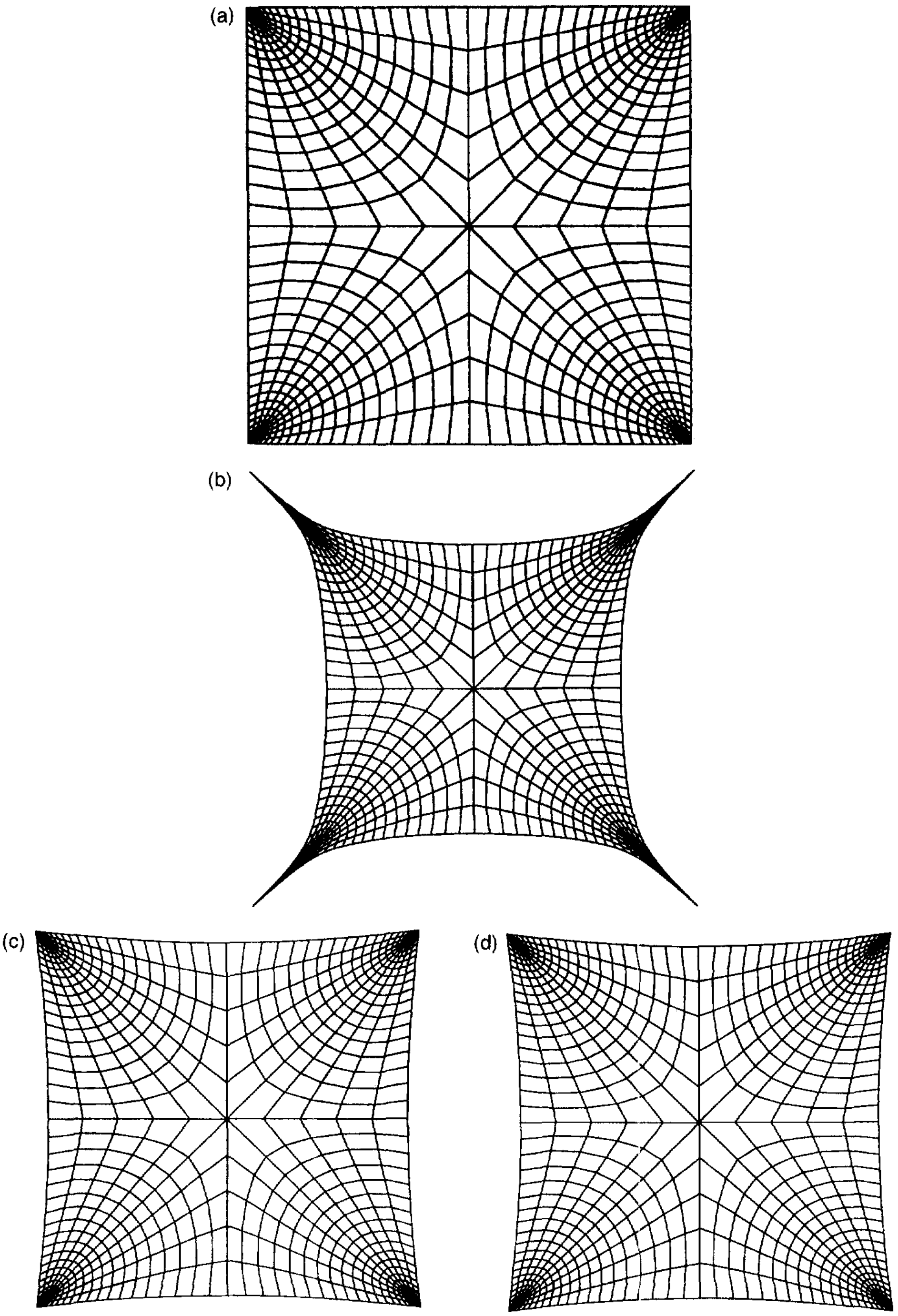

Fig. 7. Plane deformation of a square sheet of Ogden material produced by displacement of the corners: (a) reference configuration and finite difference mesh; (b) deformed mesh with traction-free edges; (c) deformed mesh with type-1 cable attached; (d) deformed mesh with type-2 cable attached. The cable modulus is $E/\mu L = 10.0$.

6.4. *Sutures*

It is intuitively plausible that the suturing of two membranes along edges that are not compatible may induce wrinkling. We exhibit this effect in two simple examples of plane deformation. The first is shown in Fig 8(a) together with a mesh consisting of 121 nodes covering each of two chevron-shaped reference configurations. The two regions are mirror images of each other, but no symmetry conditions are imposed. The perpendicular distance between the vertical edges is $10L$, where L is the length scale. The distance between

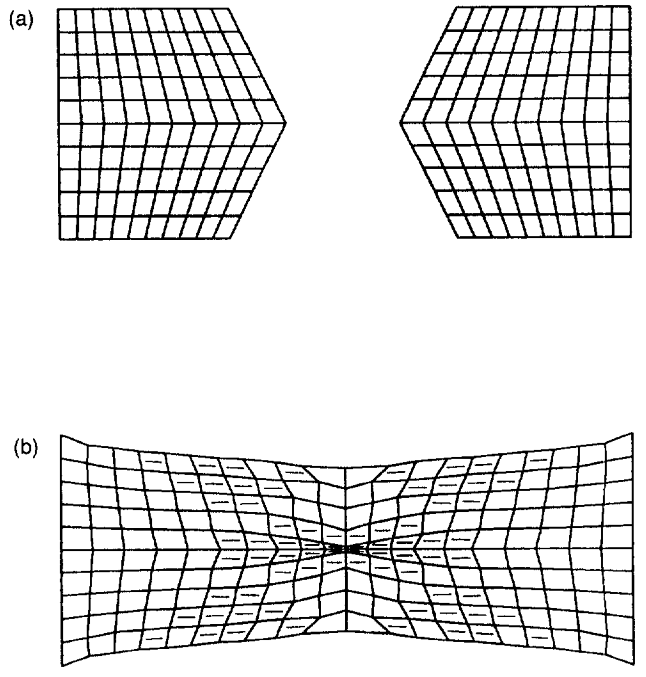

Fig. 8. Symmetric suturing of two neo-Hookean membranes: (a) reference configurations; (b) deformed configuration. Dashed lines represent trajectories of wrinkling.

horizontal edges is $4L$ and the vertex of each region is $4L$ from the corresponding vertical edge.

The two pairs of opposing oblique edges are sutured together using the procedure described in Section 5.2. The vertical and horizontal edges are fixed and traction-free, respectively, and there are no attached cables. The deformed configuration shown in Fig. 8(b) was obtained using the relaxed neo-Hookean energy defined by eqns (74) and (77). The deformation possesses the expected symmetry with a maximum computed stretch of 2.06 occurring at the zone-centered points nearest the extreme nodes at the top and bottom of the sutured edges. The stretches at all points are such that the neo-Hookean model furnishes local quantitative agreement with biaxial and uniaxial data on rubber.

Wrinkling of the membrane is indicated by the dashed lines in the interiors of the cells of the deformed mesh. They indicate the direction-fields of the local states of uniaxial tension wherever the membrane is wrinkled. These regions correspond to stretches belonging to the second branch of the domain of the function (74).

An asymmetric suturing problem is illustrated in Fig. 9(a). Here the left chevron of the previous example is replaced by a square and the opposing edges of the two regions are sutured to produce the asymmetric deformation shown in Fig. 9(b). All other prescribed conditions are as before. The largest computed stretch is now 1.61 at the zone points nearest the upper and lower right-hand corners of the square. The pattern of wrinkling induced by the suturing is also indicated.

6.5. *A three-dimensional tent*

Our final example combines suturing, wrinkling and cable–membrane interaction in a setting that relates to the practical analysis of fabric tension structures. In a typical application, several flat patches of fabric material, whose shapes are determined in accordance with some design considerations, are required to be sutured together. Certain points of the assembled membrane are then assigned particular positions in space. Subsequent tailoring of the shape and the stress distribution is effected by the stretching of one or more attached cables.

Here we present analyses of a simple model problem with all of the foregoing features. Our methods may be used to analyze the quality of actual tension structure designs, granted

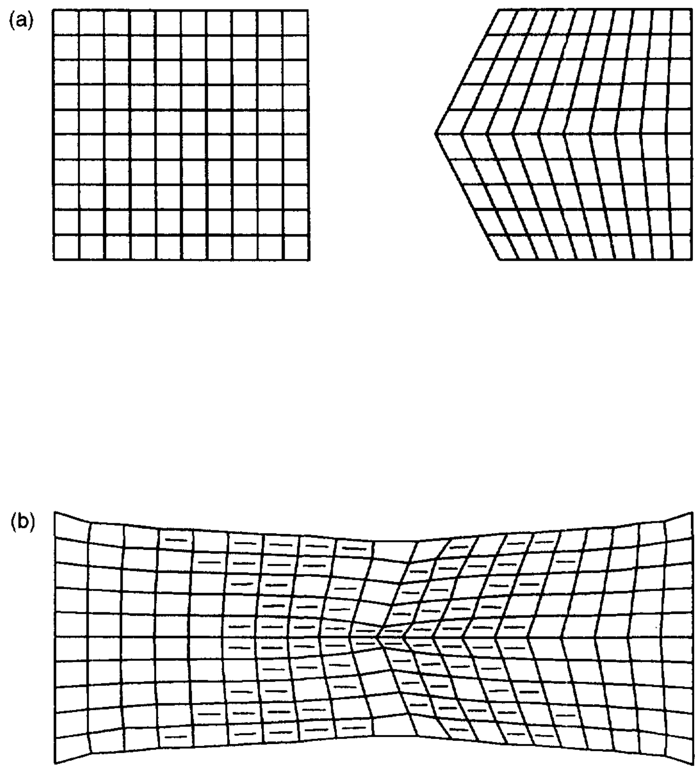

Fig. 9. Asymmetric suturing of neo-Hookean membranes: (a) reference configurations; (b) deformed configuration. Dashed lines represent wrinkle trajectories.

the availability of a suitable constitutive model for the fabric material, and a rationale for the selection of appropriate geometries of the patches to be sutured together. It is our view that the latter two issues have yet to be resolved satisfactorily. For the sake of illustration we therefore use isotropic elasticity to model the membrane material and consider only the simplest geometries for the pieces to be sutured.

Thus, consider the pair of overlapping triangles shown in Fig. 10(a). The reference configuration of the membrane structure is obtained by copying the pair of triangles shown and rotating them by successive 90° increments to cover the entire square. We impose symmetry conditions along the interior edges of the triangles aligned with the edges of the square; thus we regard the structure as being composed of four triangles, each having a base that coincides with one of the four sides of the square. The base is used as the length scale L. The oblique edges of the overlapping triangles are sutured together at corresponding nodes. The boundary value problem is then specified by fixing the triangles at the corners of the square, and requiring the common deformed position of the central vertices to be directly above the geometric center of the square, at a distance equal to one-half the base of a triangle. Cables are attached to the remaining external edges.

The meshes shown are designed to resolve the strain gradients expected near the center and the corners. Each of the two triangular regions shown contains 310 nodes. Let $\{\mathbf{e}_i\}$ be an orthonormal basis with $\mathbf{e}_1$ and $\mathbf{e}_2$ along the horizontal and vertical symmetry axes, respectively. Symmetry is imposed by requiring the components $y_2(=\mathbf{y}\cdot\mathbf{e}_2)$ and $y_1(=\mathbf{y}\cdot\mathbf{e}_1)$ of the nodal position vectors along the respective axes to vanish. The remaining components of these position vectors are computed using the dynamic relaxation algorithm (128). Thus the corresponding nodal force components vanish in equilibrium. The reaction forces generated by the position constraints are not computed, nor are they required. The vanishing of the net nodal force vectors approximates the vanishing of the divergence of the Piola stress along the symmetry axes [cf. eqn (117)]. The formulae underlying the Green's theorem difference scheme then furnish an approximation to the traction continuity that would be

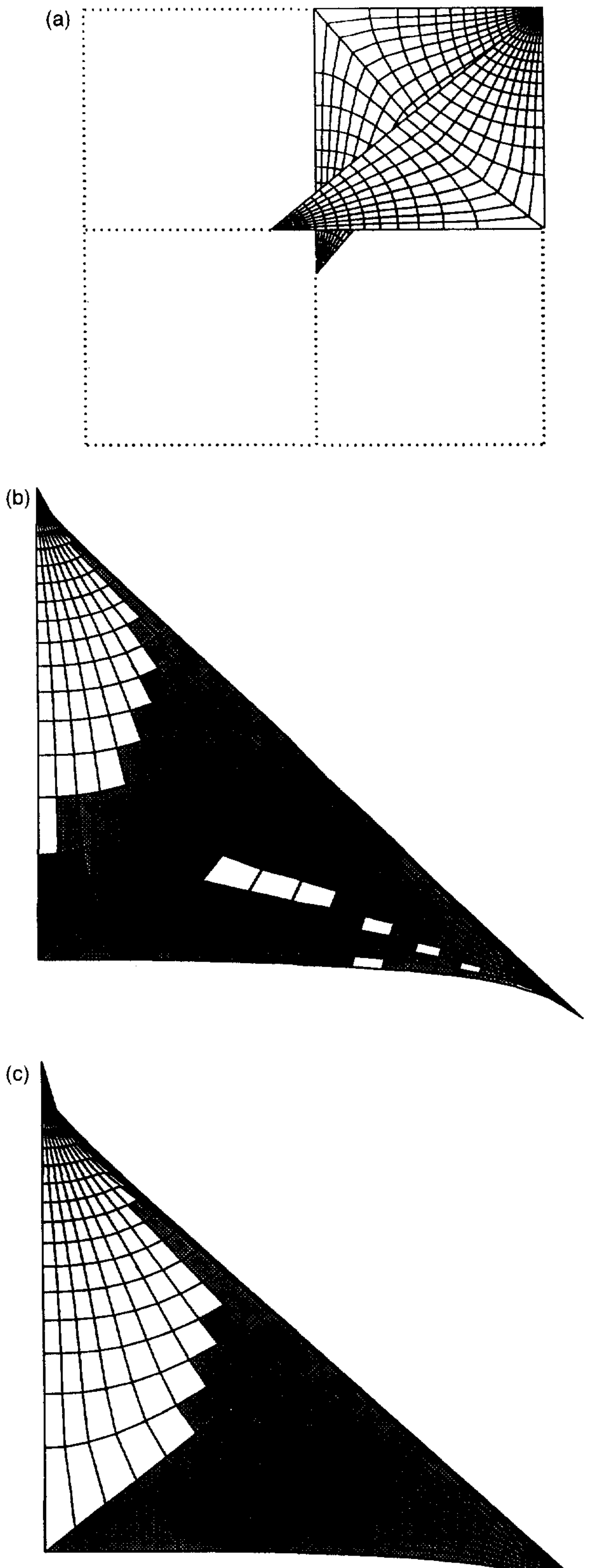

Fig. 10. Three dimensional deformation of sutured membranes composed of Ogden material: (a) reference configuration with overlapping meshes. Dotted lines indicate outer boundary and symmetry axes; (b) edge view of one quarter of the deformed membrane with traction-free edges. Light areas are biaxially stressed, shaded areas are wrinkled, and dark areas are slack (stress-free); (c) deformed membrane with type-1 cable. Light areas are biaxially stressed, shaded areas are wrinkled and slack regions do not appear. The cable modulus is $E/\mu L = 0.5$; (d) type-2 problem with the same cable modulus as in (c).

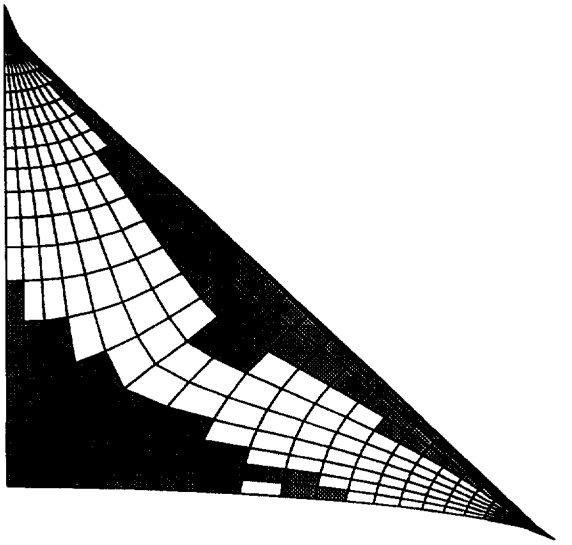

Fig. 10(d).

imposed in an exact treatment of the symmetry conditions. The constitutive equations in turn yield approximate continuity of the deformed tangent planes along the orthogonal symmetry axes of the assembled structure.

Edge-wise views of the resulting three-dimensional deformations are presented in Fig. 10(b–d). The vertical edges in the figures correspond to symmetry axes, while the lower (curved) edges are either traction-free or have cables attached. The latter curves are the deformed images of the sides of the reference square. In all cases the Ogden material and its associated relaxation are used. Figure 10(b) shows the deformed surface corresponding to traction-free edges. The light regions are under local biaxial tension while the shaded and dark regions are wrinkled and slack (stress-free), respectively. The strains in these regions belong to the second and fourth subdomains of the relaxed energy defined by eqns (74), (157) and (158). The maximum computed stretch is 2.69 at the zone-centered points nearest the four corners.

The type-1 cable–membrane problem is illustrated in Fig. 10(c). Here a cable with dimensionless modulus $E/\mu L = 0.5$ is fixed to the membrane along its lower edge. A substantial redistribution of strain is evident despite the relatively low cable stiffness. The extent of wrinkling (shaded regions) is reduced and slack regions have been eliminated. The maximum computed stretch is 2.60, but now occurs at zone-centered points nearest the apex. Finally, the type-2 problem corresponding to a freely sliding cable is shown in Fig. 10(d). The cable stiffness is unchanged, but again a pronounced strain redistribution is apparent. The maximum computed stretch is now 2.30 in the immediate vicinities of the four corners. The material adjoining these corners is biaxially stressed, whereas it is wrinkled in the preceding examples.

In actual tension structure design one of the main functions of cables is to eliminate unwanted wrinkling of the membrane by controlling the stress distribution to the extent possible. Our results give an indication of their effectiveness in this role. Another factor of considerable importance for the control of membrane stress, and thus, indirectly, membrane shape, is the design of the patterns of the various pieces of which the membrane is composed. The latter problem will be addressed in a forthcoming work.

Acknowledgements—We are grateful to the Natural Sciences and Engineering Research Council of Canada for supporting this work through Strategic Grant #STRIN 205. A. Atai gratefully acknowledges the partial support of the Iranian Ministry of Culture and Higher Education. We have also benefitted from discussions with Dr X. Li of Warner Shelter Systems Ltd, Calgary, Alberta.

REFERENCES

Acerbi, E. and Fusco, N. (1984) Semicontinuity problems in the calculus of variations. *Archive for Rational Mechanics and Analysis* **86,** 125–145.

Atai, A. and Steigmann, D. J. (1997) On the nonlinear mechanics of discrete networks. *Archive of Applied Mechanics* **67,** 303–319.

Ball, J. M. (1977) Convexity conditions and existence theorems in nonlinear elasticity. *Archive for Rational Mechanics and Analysis* **63,** 337–403.

Bliss, G. A. (1946) *Lectures on the Calculus of Variations.* University of Chicago Press.

Bufler, H. and Schneider, H. (1994) Large strain analysis of rubber-like membranes under dead weight, gas pressure, and hydrostatic loading. *Computational Mechanics* **14,** 165–188.

Cannarozzi, M. (1985) Stationary and extremum variational formulations for the elastostatics of cable networks. *Meccanica* **20,** 136–143.

Carroll, M. M. (1988) Finite strain solutions in compressible elasticity. *Journal of Elasticity* **20,** 65–92.

Como, M. and Grimaldi, A. (1995) *Theory of Stability of Continuous Elastic Structures.* CRC Press, Boca Raton, FL.

Dacarogna, B. (1989) *Direct Methods in the Calculus of Variations.* Springer, Berlin.

Fisher, D. (1988) Configuration dependent pressure potentials. *Journal of Elasticity* **19,** 77–84.

Flanagan, D. P. and Belytschko, T. (1981) A uniform strain hexahedron and quadrilateral with orthogonal hourglass control. *International Journal for Numerical Methods in Engineering* **17,** 679–706.

Gao, D. Y. (1992) Global extremum criteria for finite elasticity. *ZAMP* **43,** 924–937.

Gao, Y. and Strang, G. (1989) Geometric nonlinearity: potential energy, complementary energy, and the gap function. *Quarterly of Applied Mathematics* **XLVII,** 487–504.

Haseganu, E. M. and Steigmann, D. J. (1994a) Analysis of partly wrinkled membranes by the method of dynamic relaxation. *Computational Mechanics* **14,** 596–614.

Haseganu, E. M. and Steigmann, D. J. (1994b) Theoretical flexural response of a pressurized cylindrical membrane. *International Journal of Solids and Structures* **31,** 27–50.

Haseganu, E. M. and Steigmann, D. J. (1996) Equilibrium analysis of finitely deformed elastic networks. *Computational Mechanics* **17,** 359–373.

Herrmann, W. and Bertholf, L. D. (1983) Explicit Lagrangian finite-difference methods. *Computational Methods for Transient Analysis*, eds T. Belytschko and T. J. R. Hughes, pp. 361–416. Elsevier, Amsterdam.

Klouček, P. and Luskin, M. (1994) The computation of the dynamics of the martensitic transformation. *Continuum Mechanics and Thermodynamics* **6,** 209–240.

Knops, R. J. and Wilkes, E. W. (1973) Theory of elastic stability. *Handbuch der Physik*, Vol. VIa/3, ed. S. Flügge, p. 125. Springer, Berlin.

Kohn, R. V. and Strang, G. (1986) Optimal design and relaxation of variational problems. *Communications on Pure and Applied Mathematics* **39,** 1–25 (Part I), 139–182 (Part II) and 353–377 (Part III).

Li, X. and Steigmann, D. J. (1995) Point loads on a hemispherical elastic membrane. *International Journal of Non-Linear Mechanics* **30,** 569–581.

Libai, A. and Simmonds, J. G. (1997) *The Nonlinear Theory of Elastic Shells*, 2nd edn. Cambridge University Press. In press.

Mansfield, E. (1953) Neutral holes in plane sheet-reinforced holes which are elastically equivalent to the uncut sheet. *Quarterly Journal of Mechanics and Applied Mathematics* **6,** 370–378.

Naghdi, P. M. and Tang, P. Y. (1977) Large deformation possible in every isotropic elastic membrane. *Philosophical Transactions of the Royal Society of London* **A287,** 145–187.

Odgen, R. W. (1984) *Non-linear Elastic Deformations.* Ellis-Horwood, Chichester.

Papadrakakis, M. (1981) A method for the automatic evaluation of dynamic relaxation parameters. *Computational Methods in Applied Mechanics and Engineering* **25,** 35–48.

Pipkin, A. C. (1986) The relaxed energy density for isotropic elastic membranes. *IMA Journal of Applied Mathematics* **36,** 85–99.

Pipkin, A. C. (1994) Relaxed energy densities for large deformations of membranes. *IMA Journal of Applied Mathematics* **52,** 297–308.

Podio-Guidugli, P. (1988) A variational approach for live loadings in finite elasticity. *Journal of Elasticity* **19,** 25–36.

Rybka, P. (1992) Dynamical modeling of phase transitions by means of viscoelasticity in many dimensions. *Proceedings of the Royal Society of Edinburgh* **121A,** 101–138.

Silling, S. A. (1987) Incompressibility in dynamic relaxation. *ASME Journal of Applied Mechanics* **54,** 539–544.

Silling, S. A. (1988a) Numerical studies of loss of ellipticity near singularities in an elastic material. *Journal of Elasticity* **19,** 213–239.

Silling, S. A. (1988b) Finite difference modelling of phase changes and localization in elasticity. *Computational Methods in Applied Mechanics and Engineering* **70,** 251–273.

Silling, S. A. (1989) Phase changes induced by deformation in isothermal elastic crystals. *Journal of the Mechanics and Physics of Solids* **37,** 293–316.

Steigmann, D. J. (1986) Proof of a conjecture in elastic membrane theory. *ASME Journal of Applied Mechanics* **53,** 595–596.

Steigmann, D. J. (1991) A note on pressure potentials. *Journal of Elasticity* **26,** 87–93.

Steigmann, D. J. and Li, D. (1995) Energy-minimizing states of capillary systems with bulk, surface, and line phases. *IMA Journal of Applied Mathematics* **55,** 1–17.

Steigmann, D. J. and Ogden, R. W. (1997) Plane deformations of elastic solids with intrinsic boundary elasticity. *Proceedings of the Royal Society of London* **A453,** 853–877.

Swart, P. J. and Holmes, P. J. (1992) Energy minimization and the formation of microstructure in dynamic antiplane shear. *Archive for Rational Mechanics and Analysis* **121,** 37–85.

Tabarrok, B. and Qin, Z. (1992) Nonlinear analysis of tension structures. *Computers and Structures* **45,** 973–984.

Underwood, P. (1983) Dynamic relaxation. *Computational Methods for Transient Analysis*, eds T. Belytschko and T. J. R. Hughes, pp. 245–265. Elsevier, Amsterdam.

van Tiel, J. (1984) *Convex Analysis*. Wiley, New York.

Varga, O. H. (1966) *Stress Strain Behavior of Elastic Materials*. Wiley, New York.

Varley, E. and Cumberbatch, E. (1980) Finite deformations of elastic materials surrounding cylindrical holes. *Journal of Elasticity* **10**, 341–405.

Wu, C.-H. (1979) Large finite strain membrane problems. *Quarterly of Applied Mathematics* 347–359.